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MPIDR WORKING PAPER WP 2011-004
JANUARY 2011

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Abstract

Statistical analysis of data on the longest living humans leaves room for speculation whether the human force of mortality is actually leveling off. Based on this uncertainty, we study a mixture failure model, introduced by Finkelstein and Esaulova (2006) that generalizes, among others, the proportional hazards and accelerated failure time models. In this paper we first, extend the Abelian theorem of these authors to mixing distributions, whose densities are functions of regular variation. In addition, taking into account the asymptotic behavior of the mixture hazard rate prescribed by this Abelian theorem, we prove three Tauberian-type theorems that describe the class of admissible mixing distributions. We illustrate our findings with examples of popular mixing distributions that are used to model unobserved heterogeneity.

Keywords: mortality asymptotics, mixture survival models, frailty distributions, functions of regular variation, Tauberian theorems

1 Introduction

The International Database of Longevity IDL (2010) offers detailed information on thoroughly validated cases of supercentenarians. Gampe (2010) has used these data to estimate the human force of mortality after age 110. Her analysis revealed that human mortality between ages 110 and 114 levels off regardless of gender. It is flat at a level corresponding approximately to a 50% annual probability of death (Gampe 2010; Robine et al. 2005). As human populations are heterogeneous, this finding raises an important question, which this article addresses: what is the underlying heterogeneity model and how is individual frailty distributed if i) human mortality approaches a constant limit, or ii) mortality abandons the plateau at later ages, where there are still no officially recorded survivors, i.e. if the asymptotic behavior of the force of mortality is not constant? We address this problem in much more general settings, not restricting ourselves to a Gompertz baseline distribution and an asymptotically flat force of mortality. In fact, for a rather general frailty survival model (which includes as special cases proportional hazards and accelerated life models) and given asymptotic behavior of the hazard rate (e.g., flat force of mortality at infinity), we describe the class of admissible frailty distributions that "generates" this behavior.

A similar inverse problem was studied by Steinsaltz and Wachter (2006). It was restricted, though, to the special case of the proportional hazards frailty model. Assuming that the baseline hazard is asymptotically equivalent to a Gompertz curve and the frailty (mixing) distribution behaves in a neighborhood of zero like a power function $c\,z^{\alpha}$, where $c \equiv const$ and $\alpha > -1$, these authors prove an Abelian theorem that the resulting mixture (population) hazard rate is asymptotically flat. Finkelstein and Esaulova (2006) assume the same behavior of the frailty distribution for $z \to 0$, but for a more general survival model. They derive independently the asymptotic result of Steinsaltz and Wachter (2006) and, moreover, prove that the mixture hazard rate for the accelerated life model (ALM) tends to zero with time, regardless of the baseline lifetime distribution. This implies that if human mortality is asymptotically flat, then the underlying model is not ALM.

Steinsaltz and Wachter (2006) also proved a Tauberian theorem for the proportional hazards model, i.e., assuming that the mixture hazard rate is asymptotically flat and the underlying mortality distribution follows the Gompertz (or asymptotically Gompertz as $t \to \infty$) law, they described the set of frailty distributions that could produce this leveling-off. Thus, Steinsaltz and Wachter (2006) answer our question i) for the special case of proportional hazards.

If the proportional hazards model produces a mortality plateau and the accelerated life model results in a mortality rate that approaches zero, can we conclude that the observed flat mortality at oldest-old ages results necessarily from proportional hazards? In general, no because Abelian theorems do not provide information about the speed of convergence of the mixture failure rate to its asymptotic value. Thus, we are not formally sure whether the plateau after the age of 110 is the eventual leveling-off of the force of mortality (which might well be the case), or it is merely a constancy interval that could be followed, for instance, by an eventual decrease to zero. Therefore, the uncertainty in the asymptotic behavior of the force of mortality requires the study of mixture models with flexible (depending on model parameters) asymptotics.

In this paper we, first, generalize the results of Finkelstein and Esaulova (2006) for an important wider class of frailty distributions. Second, given the asymptotic behavior of the mixture hazard rate, we derive simple sufficient conditions about the form of the corresponding distribution of frailty. These general results could hopefully contribute to the better understanding of oldest-old human mortality patterns like, for example, the observed special case of asymptotically flat mortality.

2 Preliminaries

Conventional survival analysis models incorporate a baseline mortality law, a term that accounts for observed heterogeneity (usually a linear predictor of covariates), and a scheme by which these two are linked together. For example, factors may affect individual hazard multiplicatively, thus producing the proportional hazards model. They may instead preserve the same mortality pattern for everyone, but assign individual-specific time scaling, thus producing the ALM. These and other models can be extended to account for unobserved heterogeneity by introducing a random variable $Z \geq 0$, called *frailty*, that captures individual-specific susceptibility to experiencing the event of interest (in demography, usually death, Vaupel et al. 1979). In the proportional hazards settings, Z is a multiplicative factor acting on individual hazard, which means that the larger the realization of Z (i.e. the "frailer" an individual), the larger the corresponding hazard rate. Our paper focuses on a wider class of models, introduced by Finkelstein and Esaulova (2006) and thoroughly studied in Finkelstein (2008), which includes as special cases the two most commonly used survival models in demography, epidemiology, medicine, biology, and engineering – the proportional hazards model and the accelerated life model (ALM). In order to proceed with our findings, we must first briefly present and discuss the relevant results in Finkelstein and Esaulova (2006).

Let $T \geq 0$ be a lifetime random variable characterized by a survival function $S(t)$. Suppose $S(t)$ is indexed (conditioned) by a random variable $Z \geq 0$ (frailty) with a pdf $\pi(z)$:

$$S(t,z) := P(T > t \,|\, Z = z) \equiv P(T > t \,|\, z),$$

where $P(A)$ denotes the probability of event A.

Suppose the pdf $f(t,z) = -S'_t(t,z)$ exists and denote the corresponding hazard rate by $\mu(t,z)$:

$$\mu(t,z) = \frac{f(t,z)}{S(t,z)}.$$

Then the mixture survival function and the pdf, i.e., the survival and the density functions of the population, are

$$S_m(t) = \int_0^\infty S(t,z)\pi(z)dz\,, \qquad f_m(t) = \int_0^\infty f(t,z)\pi(z)dz,$$

respectively. As a result, the mixture failure rate, i.e. the hazard rate of the population, is

$$\mu_m(t) = \frac{\int_0^\infty f(t,z)\pi(z)dz}{\int_0^\infty S(t,z)\pi(z)dz}.$$

Assume that the mixing distribution's pdf $\pi(z), z \geq 0$, belongs to the family defined as

$$\pi(z) = z^\alpha \pi_1(z), \tag{1}$$

where $\alpha > -1$, and the function $\pi_1(z)$ is (i) bounded in $[0, +\infty)$, (ii) continuous and nonvanishing at $z = 0$. These densities represent a product of a power term and a function that is constant at 0.

Assume that the failure distribution is characterized by a cumulative hazard

$$H(t,z) = \int_0^t \mu(x,z)dx = A(z\phi(t)). \qquad (2)$$

As the cumulative hazard $H(t,z)$ is always a differentiable non-decreasing function equal to zero at $t=0$, the functions $A(\cdot)$ and $\phi(\cdot)$ should be non-decreasing, differentiable, and such that $A(z\phi(0)) = 0$. Model (2), defined at the level of the cumulative hazard rather than the hazard rate itself, generalizes many standard models. For instance, when $A(s) \equiv s$ and $\phi(t) = H(t)$, it reduces to proportional hazards. If $A(s) = H(s)$ and $\phi(t) \equiv t$, then (2) is equivalent to the ALM. Note that, (2) can be trivially adjusted by an additive term to account for additive hazards and related models (Finkelstein 2008).

Let us assume in addition that $A(\cdot)$ and $\phi(\cdot)$ are strictly increasing and $\lim_{s\to+\infty} A(s) = +\infty$ and $\lim_{t\to+\infty} \phi(t) = +\infty$. For $e^{-A(z\phi(t))}$, which is the survival function of the mixture lifetime distribution, we assume that the following (rather weak) condition holds:

$$\int_0^\infty e^{-A(s)} s^\alpha \, ds < \infty, \qquad (3)$$

which means that the mixture lifetime distribution is not "too heavy-tailed".

Assuming (1), (2), and (3), Finkelstein and Esaulova (2006) prove that the population's force of mortality $\mu_m(t)$ has the following asymptotic behavior

$$\mu_m(t) \sim (\alpha+1)\,\frac{\phi'(t)}{\phi(t)} \qquad t \to \infty, \qquad (4)$$

where $a(t) \sim b(t)$ denotes $\lim_{t\to\infty} a(t)/b(t) = 1$. Eq. (4) means that asymptotic behavior of $\mu_m(t)$ depends only on α and the derivative of the logarithm of the scaling function $\phi(t)$. Thus, for the Gompertz proportional hazards model

$$A(s) \equiv s\,, \qquad \phi(t) = H(t) = \frac{a}{b}\,(e^{bt} - 1)\,,$$

the mixture failure rate tends to a constant:

$$\mu_m(t) \sim (\alpha+1)b \equiv const.$$

Note that, this result is true for any mortality distribution such that (Steinsaltz and Wachter 2006)

$$\lim_{t \to \infty} \frac{\mu(t)}{H(t)} = b.$$

Further in this paper we will refer to these distributions as "Gompertz-like".

In this paper we show, first, that the Abelian theorem proved in Finkelstein and Esaulova (2006) holds not only for frailty densities (1), but also for any pdf that is a product of z^α and a function of regular variation with power α larger than -1. Then, based on (4), we prove the inverse result stating that the class of frailty distributions is also the family of functions of regular variation for $z \to 0$ with power larger than -1. Finally, we consider (as simple examples) a number of widely used frailty distributions, among which the gamma, the log-normal, and the inverse Gaussian, and check whether they belong to this family.

3 Mixing Distributions and Functions of Regular Variation

We adopt the following definitions Feller (1971) for functions of slow and regular variation at 0 (see also Bingham et al. 1989).

Definition 1. *A positive function $G(t)$ defined on $(0, \infty)$ is slowly varying at 0 if for every fixed $k > 0$:*

$$\lim_{t \to 0} \frac{G(kt)}{G(t)} = 1.$$

Definition 2. *A positive function $F(t)$ defined on $(0, \infty)$ is regularly varying at 0 with power (exponent) $-\infty < p < \infty$, if*

$$\lim_{t \to 0} \frac{F(t)}{t^p \, G(t)} = 1.$$

As far as we know, only a few papers relate these functions to mixture models. For example, in the special case of a mixture of exponential distributions Abbring and van den Berg (2007) prove that if "proportional frailty" Z

is regularly varying at 0, then the random variable Zt converges in distribution to the gamma distribution with parameters 1 and p (see also Block and Joe 1997). We will use the idea of regular variation for $z \to 0$ to generalize the Abelian theorem of Finkelstein and Esaulova (2006).

As asymptotic relationship (4) depends on the mixing (frailty) distribution just in terms of its power characteristic α in a neighborhood of zero, the definitions above suggest that (4) can be valid for a wider than (1) class of mixing distributions with a pdf

$$\pi(z) = z^{\alpha}\, G(z)\, \pi_1(z), \tag{5}$$

where $G(z)$ is a slowly varying at 0 function. In fact, instead of $G(z)\, \pi_1(z)$ we can assume, in general, any regularly varying function with power α, but in view of Definition 2 and relationship (1), we consider in this section frailty with density (5). The proof in Finkelstein and Esaulova (2006) can be generalized to account for the extra multiplicative term $G(z)$. Thus, the following extension to the Abelian theorem for the general mixture model (2) holds:

Theorem 1. *Let the cumulative hazard $H(t, z)$ of a mixture failure model be given by (2) and the pdf of frailty Z be*

$$\pi(z) = z^{\alpha}\, G(z)\, \pi_1(z),$$

where $\alpha > -1$, $G(z)$ is a slowly varying at 0 function, and $\pi_1(z)$, $\pi_1(0) \neq 0$, is a bounded in $[0, \infty)$ and continuous at $z = 0$ function.
Assume that the survival function of the mixture lifetime distribution satisfies (3) and, in addition

$$\lim_{s \to \infty} A(s) = \infty \qquad \text{and} \qquad \lim_{t \to \infty} \phi(t) = \infty.$$

Then

$$\mu_m(t) \sim (\alpha + 1)\, \frac{\phi'(t)}{\phi(t)}.$$

The proof of Theorem 1 could be found in the Appendix. The gamma distribution is an example of a frailty distribution, widely used in various applications, that satisfies (5). On the other hand, it could be easily shown that (5) does not hold for other practically used frailty distributions like the inverse Gaussian and the log-normal.

The class of frailty distributions defined by (5) contains also, e.g., $\pi(z) = z^\alpha \ln(1/z)$, as $\ln(1/z)$ is slowly varying for $z \to 0$. This function was considered by Steinsaltz and Wachter (2006) as an example of a pdf that is admissible according to the Tauberian theorem for proportional hazards mixture models, but not belonging to the family of densities (1) in its Abelian counterpart. Thus, Theorem 1 extends the class of admissible frailty distributions at least by functions of the type discussed in Steinsaltz and Wachter (2006).

4 "Tauberian" Results for the Mixture Failure Rate

The mortality plateau observed in Gampe (2010) is independent of gender and time trends in supercentenarian mortality between earlier and later cohorts. As univariate frailty models are not identifiable in the absence of covariates (Elbers and Ridder 1982; Heckmann and Singer 1984; Hoem 1990; Yashin et al. 1994), we have to specify first, implicitly or explicitly, the underlying mortality distribution in order to describe the mixing distribution. We will assume that the cumulative hazard rate for individuals with frailty $Z = z$ is given by (2). Then a class of frailty distributions that produce a mixture hazard rate with asymptotics (4) is given by the following

Theorem 2. *Let the cumulative hazard rate $H(t, z)$ be given by (2) and $\lim\limits_{t\to\infty} \phi(t) = \infty$, $\lim\limits_{s\to\infty} A(s) = \infty$. Suppose that the mixture failure rate $\mu_m(t)$ satisfies*

$$\mu_m(t) \sim c\,\frac{\phi'(t)}{\phi(t)} > 0 \qquad t \to \infty,$$

where $c > 0$.
Then the pdf $\pi(z)$ of the mixing (frailty) distribution satisfies for $z \to 0$

$$\frac{\int\limits_0^\infty e^{-A(z\,\phi(t))}\, z\,\pi'(z)\,dz}{\int\limits_0^\infty e^{-A(z\,\phi(t))}\,\pi(z)\,dz} \sim c - 1. \tag{6}$$

The proof of Theorem 2 is presented in the Appendix. Since $\phi(t) \to \infty$, only the behavior of the integrands for values of z close to zero define the asymptotics both for the numerator and the denominator. Therefore, if we, for example, construct a pdf that is defined for sufficiently small z as

$$\pi(z) = Cz^{c-1}\,, \tag{7}$$

where $C > 0$ is a constant, then (7) will be a trivial solution to (6). We will show at the end of this section that regularly varying probability density functions, which can be always represented as (Bingham et al. 1989):

$$\pi(z) = z^{c-1}\,G(z)\,, \tag{8}$$

where $G(z)$ is a slowly varying at $z \to 0$ function, also satisfy (6). Note that, in general, densities (8) do not satisfy the following relationship:

$$z\,\pi'(z) = (c-1)\,\pi(z) \qquad z \to 0,$$

which can be clearly illustrated in the case of $0 < c < 1$ (i.e., when $\pi(z)$ tends to infinity as $z \to 0$) and $G(z) = \log(1/z)$ (see also Theorem 3).

The Tauberian theorem in Steinsaltz and Wachter (2006) is a special case of Theorem 2 in the sense that the admissible densities in their paper are in fact regularly varying functions given by (8), although the authors do not express them in these terms. The proof in Steinsaltz and Wachter (2006) is based, however, on the asymptotic properties of the Laplace transform, which plays an important role in proportional hazards models: the mixture survival function $S_m(t)$ is the Laplace transform of the mixing distribution, calculated for the baseline hazard $H(t)$ (Hougaard 1986). In (2), however, the mixture failure rate $\mu_m(t)$ cannot be expressed, in general, in terms of the Laplace transform. That is why the proof of Theorem 2 is based solely on the properties of limits and integrals.

Theorem 2 is in a sense an inverse (Tauberian) theorem to the Abelian theorem of Finkelstein and Esaulova (2006) and to our generalization Theorem 1. It does not use condition (3), which is not surprising because Tauberian theorems are by default "weaker" than their Abelian counterparts.

Condition (6) is given in asymptotic terms. As a result, it is difficult to describe explicitly the class of admissible frailty distributions within the framework of model (2). Nevertheless, it can be shown that certain functions of regular variation belong to this class. We will prove first the following

Theorem 3. *Suppose the assumptions of Theorem 2 hold and, in addition, the pdf* $\pi(z)$ *satisfies*

$$\lim_{z \to 0} \frac{z\,\pi'(z)}{\pi(z)} = c - 1 \tag{9}$$

Then relationship (6) holds.

Assumption (9) provides a convenient criterion for checking the admissibility of $\pi(z)$. The following theorem simplifies this procedure even more.

Theorem 4. *Let*

1. $\pi(z)$ *be a regularly varying at* 0 *function defined by (8), where* $c > 0$.
2. $\pi'(z)$ *be asymptotically monotone as* $z \to 0$.

Then relationship (9) holds.

Note: Assumption 2. in Theorem 4 can be substituted alternatively by: let the derivative of the slowly varying function $G(z)$ be asymptotically monotone as $z \to 0$.

The proofs of Theorem 3 and Theorem 4 are given in the Appendix. Note that, the assumption of monotonicity at 0 for slowly varying functions is absolutely non-restrictive in our context as only some bizarre functions, e.g., $\sin(1/z)$, do not satisfy this condition (see the next section). The function $G(z) = \log(1/z)$ mentioned above is obviously asymptotically monotone for $z \to 0$.

5 Examples of Mixing Distributions

In this section, just for a simple illustration, we will examine several popular mixing distributions for modelling frailty – the gamma (Vaupel et al. 1979), the log-normal (McGilchrist and Aisbett 1991), the inverse Gaussian (Hougaard 1984), as well as the beta and the Weibull distributions that are less commonly used. For each of them we will check whether its density satisfies the sufficient condition (9) of Theorem 3. Thus, we can classify the distributions mentioned above into two groups: "admissible" and "non-admissible" within the general framework (2).

5.1 "Admissible" Frailty Distributions

The Gamma Distribution

The gamma distribution with positive parameters k and λ is the most popular distribution for modelling frailty, especially in proportional hazards models, due to the convenient form of its Laplace transform. The gamma distribution was first used in demography for modeling heterogeneous populations in Beard (1959), and was later introduced by Vaupel et al. (1979) for studying univariate frailty models. The density of the gamma distribution

$$f_\Gamma(z; \lambda, k) = \frac{\lambda^k}{\Gamma(k)}\, z^{k-1}\, e^{-\lambda z}$$

satisfies (9) for $k = c$. Indeed,

$$z\,\pi'(z) = \pi(z)\,(k - 1 - \lambda z)$$

and that is why

$$\lim_{z \to 0} \frac{z\,\pi'(z)}{\pi(z)} = k - 1.$$

As a result, the gamma distribution is a plausible mixing distribution for the general model (2). We can prove this also by checking the necessary conditions of Theorem 4: the gamma density satisfies (8) with $k = c$, the function $G(z) = \lambda^k\, e^{-\lambda z}/\Gamma(k)$ is slowly varying at 0, and its derivative is asymptotically ($z \to 0$) monotone.

The Weibull Distribution

Although not frequently used as a frailty (but rather as a baseline failure) distribution, the Weibull distribution with parameters $a > 0$ and $b > 0$ is also admissible in terms of (9). Its density

$$\pi(z) = f_{\text{Weibull}}(z; a, b) = \frac{a}{b}\left(\frac{z}{b}\right)^{a-1} e^{-\left(\frac{z}{b}\right)^a} \tag{10}$$

implies that

$$z\,\pi'(z) = \pi(z)\left(a - 1 - \frac{a}{b^a}\,z^a\right)$$

and, as a result,

$$\lim_{z\to 0} \frac{z\,\pi'(z)}{\pi(z)} = a - 1$$

The Beta Distribution

The beta distribution with parameters $a > 0$ and $b > 0$ is sometimes used as an alternative to the gamma, when frailty is quantified in $(0, 1)$ rather than $(0, \infty)$. The beta density is given by

$$\pi(z) = f_{\text{Beta}}(z; a, b) = \frac{z^{a-1}(1-z)^{b-1}}{B(a, b)}, \tag{11}$$

where $B(a, b) = \int_0^1 x^{a-1}\,(1-x)^{b-1}\,dx$ is the beta function. Taking advantage of

$$z\,\pi'(z) = \pi(z)\left(a - 1 - \frac{b-1}{1-z}\,z^a\right),$$

we can see that (9) is fulfilled for $a = c$. Alternatively, we can prove that the beta distribution is admissible by applying Theorem 4.

5.2 "Non-Admissible" Frailty Distributions

The Log-Normal Distribution

The log-normal distribution with a location parameter $m \in \mathbb{R}$ and a squared scale parameter $\sigma^2 > 0$, used in survival models among others in McGilchrist and Aisbett (1991), has a density

$$\pi(z) = f_{\text{logN}}(z; m, \sigma^2) = \frac{1}{z\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln z - m)^2}{2\sigma^2}\right\},$$

which implies

$$z\,\pi'(z) = \pi(z)\left(-1 - \frac{\ln z - m}{\sigma^2}\right).$$

In this case (9) does not hold as $\lim_{z\to 0} \ln z = -\infty$. As a result, the log-normal distribution cannot be picked up as a mixing distribution in the framework of (2). This can be checked also in terms of the criterion in Theorem 4. If we try to express the log-normal density in terms of (8), we will have

$$f_{\mathrm{logN}}(z; m, \sigma^2) = z^{c-1}\,G(z), \qquad G(z) = \frac{1}{z^c \sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln z - m)^2}{2\sigma^2}\right\}.$$

However, this $G(z)$ is not slowly varying at 0, and neither is its derivative monotone for $z \to 0$.

The Inverse Gaussian Distribution

The inverse Gaussian distribution with parameters $\mu, \lambda > 0$ has a pdf

$$\pi(z) = f_{\mathrm{InvGauss}}(z; \mu, \lambda) = \sqrt{\frac{\lambda}{2\pi z^3}} \exp\left\{-\frac{\lambda(z-\lambda)^2}{2\mu^2 z}\right\},$$

which yields

$$z\,\pi'(z) = \pi(z)\left(-\frac{3}{2} - \frac{\lambda}{2\mu}\,z + \frac{\lambda^3}{2\mu z}\right).$$

Due to the last term in the parentheses, which tends to infinity as $z \to 0$, (9) does not hold. Therefore, the inverse Gaussian distribution is also excluded from the class of plausible distributions for the general model (2).

6 Conclusion

This paper aims at answering the question what distributions we can use for frailty if mortality has certain asymptotic behavior. We study a general mixture model, proposed by Finkelstein and Esaulova (2006), which includes as special cases the proportional hazards and the accelerated failure time models. The latter cannot produce a plateau as its mixture hazard rate tends to zero. As a result, if mortality gets flat at oldest-old ages, the underlying model can be proportional hazards or some other, excluding the accelerated failure time model. In the case of proportional hazards, the mortality distribution is "Gompertz-like" and the frailty distribution is given either as in Steinsaltz and Wachter (2006), or by (6). If the model is not proportional hazards, then we can still classify the plausible mixing distributions by (6) or (9). Theorem 4 offers a suitable sufficient condition for checking whether a distribution belongs to a subset of the "plausible" class. Among the popular distributions used to describe frailty, the ones that satisfy (6) are the gamma, beta, and Weibull distribution.

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Appendix: Proofs of Theorems 1 – 4

Proof of Theorem 1:
▷ In the proof of Theorem 1 we will take advantage of Lemma 1 in Finkelstein and Esaulova (2006): if $g(z)$ and $h(z)$ are two nonnegative locally integrable in $[0, \infty)$ functions such that

$$\int_0^\infty g(z)\, dz < \infty$$

and $h(z)$ is bounded and continuous at $z = 0$, then

$$\lim_{t\to\infty} t \int_0^\infty g(tz)\, h(t)\, dz = h(0) \int_0^\infty g(z)\, dz.$$

We will further refer to this result by Lemma 1.
By definition the mixture hazard rate $\mu_m(t)$ is

$$\mu_m(t) \quad = \quad \frac{\int_0^\infty f(t,z)\, \pi(z)\, dz}{\int_0^\infty S(t,z)\, \pi(z)\, dz}. \tag{12}$$

Let us first derive an asymptotic expression for the denominator of (12) when $t \to \infty$. Using model specification (2), we can rewrite it as

$$\int_0^\infty S(t,z)\, \pi(z)\, dz \quad = \quad \int_0^\infty e^{-A(z\,\phi(t))}\, z^\alpha\, G(z)\, \pi_1(z)\, dz.$$

Performing the substitution $s = z\,\phi(t)$ in the integrand, we get

$$\int_0^\infty S(t,z)\, \pi(z)\, dz \quad = \quad \frac{1}{[\phi(t)]^{\alpha+1}} \int_0^\infty e^{-A(s)}\, s^\alpha\, G\left(\frac{s}{\phi(t)}\right)\, \pi_1\left(\frac{s}{\phi(t)}\right)\, ds$$

Applying Lemma 1 for $g(s) = e^{-A(s)} s^\alpha$ and $h(s) = \pi_1(s)$, we get for $t \to \infty$

$$\int_0^\infty S(t,z)\,\pi(z)\,dz \sim \frac{\pi_1(0)}{[\phi(t)]^{\alpha+1}} \int_0^\infty e^{-A(s)}\, s^\alpha\, G\left(\frac{s}{\phi(t)}\right) dz \sim \tag{13}$$

$$\sim \frac{\pi_1(0)}{[\phi(t)]^{\alpha+1}}\, G\left(\frac{1}{\phi(t)}\right) \int_0^\infty e^{-A(s)}\, s^\alpha\, dz. \tag{14}$$

The last step in these derivations is due to Definition 1. Formally, this operation is "not uniform" with respect to s. However, since the integral on the right-hand side is converging (see (3)), everything can be easily justified by dividing the interval of integration into two parts and performing the respective operations.

In a similar way, taking advantage of the fact that

$$f(t,z) = z\,\phi'(t)\,A'(z\,\phi(t))\,e^{-A(z\,\phi(t))},$$

and again using Lemma 1 and the property of the slowly varying function, we can express asymptotics for $t \to \infty$ of the numerator in (12) as

$$\int_0^\infty f(t,z)\,\pi(z)\,dz \sim \frac{(\alpha+1)\phi'(t)\,\pi_1(0)}{[\phi(t)]^{\alpha+2}}\, G\left(\frac{1}{\phi(t)}\right) \int_0^\infty e^{-A(s)}\, s^\alpha\, dz. \tag{15}$$

Substituting (14) and (15) in (12) results in asymptotic relationship (4). Q.E.D. $\lhd$

Proof of Theorem 2:

$\rhd$ As shown in Appendix A, the mixture failure rate $\mu_m(t)$ in the settings of model (2) can be expressed as a ratio of two integrals:

$$\mu_m(t) = \frac{\phi'(t) \int_0^\infty e^{-A(z\phi(t))}\, A'(z\phi(t))\, z\,\pi(z)\,dz}{\int_0^\infty e^{-A(z\phi(t))}\pi(z)\,dz}. \tag{16}$$

After making a substitution $s = z\,\phi(t)$ and integrating the result by parts, we express the integral in the numerator of (16) as

$$
\begin{aligned}
- \quad & \phi'(t) \int_0^\infty e^{-A(z\phi(t))}\, A'(z\phi(t))\, z\, \pi(z)\, dz = -\frac{\phi'(t)}{\phi^2(t)} \int_0^\infty e^{-A(s)}\, A'(s)\, s\, \pi\left(\frac{s}{\phi(t)}\right)\, ds = \\
= \quad & \frac{\phi'(t)}{\phi^2(t)} \left(\int_0^\infty e^{-A(s)}\, \pi\left(\frac{s}{\phi(t)}\right)\, ds + \frac{1}{\phi(t)} \int_0^\infty e^{-A(s)}\, s\, \pi'\left(\frac{s}{\phi(t)}\right)\, ds \right).
\end{aligned}
$$

Making the same substitution $s = z\,\phi(t)$ for the integral in the denominator of (16), we get

$$
\int_0^\infty e^{-A(z\phi(t))} \pi(z)\, dz = \frac{1}{\phi(t)} \int_0^\infty e^{-A(s)}\, \pi\left(\frac{s}{\phi(t)}\right)\, ds.
$$

Thus, the given asymptotic result for $\mu_m(t)$ $(t \to \infty)$

$$
\mu_m(t) \sim c\, \frac{\phi'(t)}{\phi(t)}
$$

can be rewritten as

$$
\frac{\frac{1}{\phi(t)} \int_0^\infty e^{-A(s)}\, s\, \pi'\left(\frac{s}{\phi(t)}\right)\, ds}{\int_0^\infty e^{-A(s)}\, \pi\left(\frac{s}{\phi(t)}\right)\, ds} \sim c - 1 \tag{17}
$$

or, using the initial variables,

$$
\frac{\int_0^\infty e^{-A(z\,\phi(t))}\, z\, \pi'(z)\, dz}{\int_0^\infty e^{-A(z\,\phi(t))}\, \pi(z)\, dz} \sim c - 1.
$$

Q.E.D. ◁

Proof of Theorem 3:
▷ Let us rewrite (9) in an equivalent and more convenient (for our current purpose) form:

$$z\,\pi'(z) = (c-1)\,\pi(z)\,[1+o(1)],$$

where, as usual, $\lim\limits_{z\to 0} o(1) = 0$. Substituting this expression for $z\,\pi'(z)$ into the left-hand side of (6), we get

$$(c-1)\,\frac{\int\limits_0^\infty e^{-A(z\,\phi(t))}\,\pi(z)\,[1+o(1)]\,dz}{\int\limits_0^\infty e^{-A(z\,\phi(t))}\,\pi(z)\,dz} \sim c-1.$$

The last step is already asymptotic with respect to $\phi(t) \to \infty$. It implies directly the statement of Theorem 3, as in this case only the behavior of the integrands for $z \to 0$ defines the asymptotics of the corresponding ratio.
Q.E.D. ◁

Proof of Theorem 4:
▷ As $\pi(z)$ is regularly varying (with power $c-1$) and $\pi'(z)$ is asymptotically (as $z \to 0$) monotone, then in accordance with the monotone density theorem Bingham et al. (1989), relationship (9) trivially holds.

Now, differentiating (8), we get

$$\pi'(z) = G'(z)\,z^{c-1} + (c-1)\,G(z)\,.z^{c-2}$$

Therefore, equation

$$\lim_{z\to 0}\frac{G'(z)\,z^{c} + (c-1)\,G(z)\,z^{c-1}}{z^{c-1}\,G(z)} = c-1$$

holds, if

$$\lim_{z \to 0} \frac{z\, G'(z)}{G(z)} = 0,$$

but the last relationship is true again due to the monotone density theorem, as $c = 1$ for the slowly varying functions. Q.E.D. $\triangleleft$