A

Vitality Model - Appendix

A.1 Solving Differential Equations

Solving for $\boldsymbol{\lambda_\psi}$

The Maximum Principle requires that

$$\dot{\lambda}_\psi \;=\; -\frac{dH}{d\psi}\;, \tag{A.1}$$

hence

$$\dot{\lambda}_\psi \;=\; -e^{-\phi}\,(1\,-\,\pi)^{\eta_r}\,\epsilon_\psi \;-\; \lambda_\psi\,(\pi^{\eta_g}\,\epsilon_\psi\;-\;\delta)\;-\;\lambda_\phi\,\mu_\psi\;. \tag{A.2}$$

Solving the differential equation leads to

$$\lambda_\psi(a) \;=\; \left(-\int_0^a g_\psi\,e^{\int_0^x f_\psi(s)\,ds}\,dx\;+\;A\right)\,e^{-\int_0^a f_\psi(s)\,ds} \tag{A.3}$$

with

$$g_\psi \;\equiv\; e^{-\phi}\,(1\,-\,\pi)^{\eta_r}\,\epsilon_\psi\;+\;\lambda_\phi\,\mu_\psi \tag{A.4}$$

capturing the change in fertility and in mortality with respect to a change in vitality, and with

$$f_\psi \;\equiv\; \pi^{\eta_g}\,\epsilon_\psi\;-\;\delta \tag{A.5}$$

capturing the change in growth with respect to a change in vitality. Note that the change in energy with respect to vitality ϵ_ψ is given by

$$\epsilon_\psi \;=\; 0.75\,k\,\psi^{-0.25}\;-\;\kappa\;, \tag{A.6}$$

being the derivative of (5.12) with respect to vitality.

Applying the transversality condition (5.20) in (A.3) one can solve for the constant A and find

$$\lambda_\psi(a) \;=\; \int_a^\infty \; g_\psi \, e^{\int_a^x \; f_\psi(s) \; ds} \; dx \; , \tag{A.7}$$

i.e.

$$\lambda_\psi(a) \;=\; \int_a^\infty \; \Big(e^{-\phi} \, (1 \,-\, \pi)^{\eta_r} \, \epsilon_\psi \,+\, \lambda_\phi \, \mu_\psi \Big) \times \; e^{\,\int_a^x \; \pi^{\eta_g} \, \epsilon_\psi \,-\, \delta \; ds} \; dx \; . \tag{A.8}$$

The shadow price of vitality at age a is given by the associated cumulated changes in fertility and mortality over all remaining ages discounted by the corresponding cumulative changes in growth.

Solving for $\boldsymbol{\lambda_\phi}$

The Maximum Principle further requires that

$$\dot{\lambda}_\phi \;=\; -\frac{dH}{d\phi} \; , \tag{A.9}$$

hence

$$\dot{\lambda}_\phi \;=\; e^{-\phi} \, (1 \,-\, \pi)^{\eta_r} \; \epsilon(\psi) \tag{A.10}$$

and thus

$$\lambda_\phi \;=\; \int_0^a \; e^{-\phi} \, (1 \,-\, \pi)^{\eta_r} \, \epsilon(\psi) \; dx \;+\; C \; . \tag{A.11}$$

Again applying the transversality conditions (5.20) helps to solve for the constant C:

$$\lambda_\phi(a) \;=\; -\int_a^\infty \; e^{-\phi} \, (1 \,-\, \pi)^{\eta_r} \, \epsilon(\psi) \; dx \; . \tag{A.12}$$

The shadow price of the cumulative hazard of death at age a is the negative value of remaining reproduction at age a, i.e. the penalty for having one unit higher cumulative hazard.

The expression in (A.12) can be substituted in (A.8) to yield the expression for $\lambda_\psi(a)$:

$$\lambda_\psi(a) \;=\; \int_a^\infty \; e^{\int_a^x \; \pi^{\eta_g} \, \epsilon_\psi \,-\, \delta \; ds} \times \Big(e^{-\phi} \, (1-\pi)^{\eta_r} \, \epsilon_\psi + \frac{b}{\psi^2} \; \int_x^\infty \; e^{-\phi} \, (1-\pi)^{\eta_r} \, \epsilon(\psi) \; d\tau \Big) \; dx \; . \tag{A.13}$$

The shadow price of vitality is given by the benefits of increasing reproduction due to higher vitality as well as the gains in remaining reproduction due to lower mortality, both weighted by the change in growth. As long as an increase in vitality leads to faster growth, this weight is above one (revaluating), if the increase in vitality leads to slower growth, then the weight is below one (devaluating).

Solving for Vitality

The differential equation in (5.14) can be solved substituting $z^4 = \psi$. After solving and re-substituting ψ, the equation for vitality is given by

$$\psi(a) = \left[\frac{k}{4}\int_0^a \pi^{\eta_g}\, e^{-\frac{\kappa}{4}\int_s^a \pi^{\eta_g}\, d\tau - \frac{\delta}{4}(a-s)}\, ds + \psi(0)^{\frac{1}{4}}\, e^{-\frac{\kappa}{4}\int_0^a \pi^{\eta_g}\, d\tau - \frac{\delta}{4} a}\right]^4 \tag{A.14}$$

where $\psi(0)$ corresponds to vitality at age zero.

Note that for $\pi = 0$ expression (A.14) simplifies to

$$\psi(a) = \psi(0)\, e^{-\delta a}\,, \tag{A.15}$$

and for $\pi = 1$ to

$$\psi(a) = \left[\frac{k}{\kappa + \delta} - e^{-0.25\,(\kappa+\delta)\,a}\left(\frac{k}{\kappa + \delta} - \psi(0)^{0.25}\right)\right]^4 . \tag{A.16}$$

A.2 Proof of Non-Existence of an Optimal Solution for a Special Case

Given that both η_r and η_g exceed one it can be proven that for the special case of constant mortality (i.e. $b = 0$) no optimal solution exists.

Proof From (5.19) it follows that

$$H(\pi = 0) = \lambda_\psi\,(\epsilon(\psi) - \delta\,\psi) + \lambda\,\mu(\psi) \tag{A.17}$$

and

$$H(\pi = 1) = e^{-\phi}\,\epsilon(\psi) - \lambda_\psi\,\delta\,\psi + \lambda\,\mu(\psi)\,. \tag{A.18}$$

Inserting those equations into the inequality $H(0) > H(1)$ and rearranging terms leads to

$$e^{\phi}\,\lambda_{\psi}\;<\;1\;. \tag{A.19}$$

The current value of the shadow price of vitality has to be smaller than one forever. Note that it is the current value of the shadow price of vitality, $\lambda_{\psi}^{c} \;\equiv\; e^{\phi}\;\lambda_{\psi}$, that matters for the optimal solution (see (5.22)).

For the special case of $\mu(\psi) \;=\; c$, i.e. $b = 0$, it can be shown that at some age a condition (A.19) will be violated: Since $\pi = 0$, (5.31) can be written as

$$\lambda_{\psi}(a) \;=\; \int_{a}^{\infty} \; e^{-c\,x}\,\epsilon_{\psi}\;e^{-\,\delta\,(x\,-\,a)}\;dx\;. \tag{A.20}$$

For constant mortality, condition (A.19) becomes $e^{c\,a}\,\lambda \;<\; 1$. Thus, multiplying (A.20) by $e^{c\,a}$ yields

$$\lambda_{\psi}^{c}(a) \;=\; \int_{a}^{\infty} \; e^{-c\,(x\,-\,a)}\,\epsilon_{\psi}\;e^{-\,\delta\,(x\,-\,a)}\;dx\;. \tag{A.21}$$

Taking into account that for $\pi = 0$ vitality is given by (A.15) and energy changes with respect to vitality according to (A.6), (A.21) becomes

$$\begin{aligned}\lambda_{\psi}^{c}(a) &= \int_{a}^{\infty} \; \epsilon_{\psi}\,e^{-(c\,+\,\delta)\,(x\,-\,a)}\;dx \\ &=\; 0.75\,\psi(a)^{-0.25}\;\frac{k}{c\,+\,0.75\,\delta}\;-\;\frac{\kappa}{c\,+\,\delta}\;.\end{aligned} \tag{A.22}$$

Does (A.19) hold? Inserting (A.22) and rearranging terms yields

$$\psi(a)\;\geq\;\left(\frac{0.75\,k\,(c\,+\,\delta)}{(\kappa\,+\,c\,+\,\delta)\,(c\,+\,0.75\,\delta)}\right)^{4}\;. \tag{A.23}$$

Since zero investment in growth ($\pi \;=\; 0$) causes vitality $\psi(a)$ to approach zero as age a approaches infinity while the right-hand side of (A.23) is a positive constant, condition (A.23) and thus condition (A.19) will definitely be violated at a certain low level of vitality.

A.3 The Algorithm

To solve the dynamic optimization problem I applied a dynamic programming approach by developing an algorithm following a backward procedure and assuming stepwise constant vitality [12]. Crucial to Bellman's approach is that the optimal decision does not depend on the past, but is based solely on the current state. The state determines possible current and future payoffs. An essential requirement for this backward optimization to work is the knowledge of an ultimate state with known payoffs, the ultimate future expectation. The procedure starts at this ultimate state and then works backwards along the state trajectory. If the mode of change can switch back and forth between growth and shrinkage, then such an ultimate state cannot be identified and the problem becomes intractable with Bellman's approach. My model constraint implies that the switch can only occur once. Since life necessarily starts off with growth, the switch is initially in up mode and can optionally change into down mode.

The state trajectory is assumed to be stepwise constant. The time it takes to change from vitality ψ to vitality $\psi \pm \Delta$ ($\Delta > 0$, step size) is given by the step time

$$\tau(\psi, \pi) = \frac{\Delta}{\dot{\psi}}, \tag{A.24}$$

where $\dot{\psi}$ is defined in Equation 5.4. Note that if vitality falls, then $\tau(\psi, \pi) = -\Delta/\dot{\psi}$ and if vitality is maintained then $\tau(\psi, \pi) = \infty$.

At each level of vitality the algorithm maximizes remaining reproduction, given by

$$R(\psi) = \int_0^{\tau} e^{-\mu(\psi)\, a}\, m(\psi, \pi)\, da + e^{-\mu(\psi)\, \tau(\psi,\pi)}\, R(\psi_{next}). \tag{A.25}$$

Since vitality is constant over the time interval τ, the integral in Equation A.25 can be solved, yielding

$$R(\psi) = \frac{m(\psi, \pi)}{\mu(\psi)} \left[1 - e^{-\mu(\psi)\tau(\psi,\pi)}\right] + e^{-\mu(\psi)\, \tau(\psi,\pi)}\, R(\psi_{next}). \tag{A.26}$$

Remaining reproduction is given by current reproduction weighted by the chance of dying in that interval and remaining reproduction at the subsequent level of vitality weighted by the probability of surviving the time interval.

The algorithm to determine the optimal investment trajectory $\pi^*(\psi)$ (the star indicates "optimal") has two parts, one for each mode. For this application, the ultimate state corresponds to a vitality of $\psi = 0$ and therefore to a mortality that is infinite and remaining reproduction of zero. Consequently, the first part of the algorithm begins in down mode at the end of possible state trajectories, i.e. at the last level of vitality $\psi > 0$ when the switch is in down mode. Since initial vitality equals one, it is convenient to choose $\psi = 1$. Then, the initial step is to find $\pi_d^*(1)$ and the corresponding $R_d^*(1)$ (the d indicates "down mode") using Equation A.26:

$$\begin{aligned}
\pi_d^*(1)) &= \max_{\pi\,\epsilon\,[0,\,\pi_0]} \; R_d(1) \qquad\qquad (A.27)\\
&= \max_{\pi\,\epsilon\,[0,\,\pi_0]} \; \frac{m(1,\pi)}{\mu(1)} \left[1 \,-\, e^{-\mu(1)\tau(1,\pi)}\right] \,+\, 0\\
&= \max_{\pi\,\epsilon\,[0,\,\pi_0]} \; \frac{(1\,-\,\pi)^{\eta_r}\,(k-\kappa)}{b+c}\\
&\qquad\qquad \times \left[1 \,-\, e^{-(b+c)\,\Delta\,/\,(\pi^{\eta_g}\,(k-\kappa)-\delta)}\right].
\end{aligned}$$

Note that my constraint implies that optimal investment π will lie between zero and π_0.

The procedure is repeated working backwards for all levels of vitality up to the maximum attainable vitality $\psi = \Psi$, determined by Equation 5.8. For each level of vitality the optimal investment is found by

$$\begin{aligned}
\pi_d^*(\psi) = \max_{\pi\epsilon[0,\,\pi_0]} \; &\frac{m(\psi,\pi)}{\mu(\psi)} \left[1 \,-\, e^{-\mu(\psi)\tau(\psi,\pi)}\right] \qquad\qquad (A.28)\\
&+ e^{-\mu(\psi)\,\tau(\psi,\pi)}\; R_d^*(\psi-\Delta).
\end{aligned}$$

This part of the algorithm gives an optimal decision for each level of vitality in down mode.

Maximum attainable vitality Ψ gives the ultimate state for the second part of the algorithm. If the switch is in up mode and vitality is at its maximum attainable level Ψ, then the decision is whether to either stay in up mode and maintain maximum vitality or to switch into down mode and follow the already calculated optimal investment in down mode:

$$\pi_u^*(\Psi) \;=\; \begin{cases} \pi_0(\Psi) \text{ if } \quad R_u^*(\Psi) \;=\; \frac{m(\Psi,\pi_0)}{\mu(\Psi)} > R_d^*(\Psi) \\[2ex] \pi_d^*(\Psi) \text{ otherwise.} \end{cases} \qquad (A.29)$$

Note that if mortality μ and fertility m are constant, then remaining reproduction is given by m/μ.

Then vitality is followed backwards, down to the smallest level of vitality $\psi = 1$. At each level of vitality the optimal investment is found by

$$\begin{aligned}\pi_u^*(\psi) &= \max_{\pi\epsilon[\pi_0,\,1]} \; R_u(\psi) \\ &= \max_{\pi\epsilon[\pi_0,\,1]} \; \frac{m(\psi,\pi)}{\mu(\psi)}\,(1 \,-\, e^{-\mu(\psi)\tau(\psi,\pi)}) \\ &\qquad\qquad + e^{-\mu(\psi)\,\tau(\psi,\pi)}\; R_u^*(\psi+\varDelta)\end{aligned} \tag{A.30}$$

if $R_u^*(\psi) > R_d^*(\psi)$ and otherwise $\pi_u^*(\psi) = \pi_d^*(\psi)$. The second part of the algorithm gives an optimal strategy for each level of vitality in up mode.

The optimal strategy over the life course can be found by connecting the results from part one and two of the algorithm in the following way: Results are saved in the form of a vector

$$\begin{pmatrix} \textit{remaining reproduction} \\ \textit{mode of change} \\ \textit{vitality} \\ \textit{investment} \\ \textit{time} \end{pmatrix} = \begin{pmatrix} R^*(\psi) \\ G,\ S \textit{ or } M \\ \psi \\ \pi^*(\psi) \\ \tau^*(\psi) \end{pmatrix} \tag{A.31}$$

Note that the variable "mode of change" takes on the value G for growth if vitality increases, S for shrinkage if vitality decreases and M for maintenance if vitality remains constant. For each level of vitality, the optimal vector is saved in a list. The optimal solution can be found from this list by connecting the vectors in the right order. The only logical succession of vectors regarding the mode of change are $(G,\ldots,G,M)$, $(G,\ldots,G,S,\ldots,S,M)$ and $(G,\ldots,G,S,\ldots,S)$. Trivially, vectors need be be nested according to subsequent levels of vitality.

Finally, the constant parameter φ can be used to adjust R^* to be equal to one. This implies that density effects produce population stationarity by reducing life-time fertility [24, 132].