Tempo and its tribulations*

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Summary. Bongaarts and Feeney offer alternatives to period life expectancy with a set of demographic measures equivalent to each other under a Proportionality Assumption. Under this assumption, we show that the measures are given by exponentially weighted moving averages of earlier values of period life expectancy. They are indices of mortality conditions in the recent past. The period life expectancy is an index of current mortality conditions. The difference is a difference between past and present, not a "tempo distortion" in the present. In contrast, the Bongaarts-Feeney tempo-adjusted Total Fertility Rate is a measure of current fertility conditions, which can be understood in terms of a process of birth-age standardization.

1 Tempo

In the study of fertility, a distinction between quantum and tempo in the spirit of Norman Ryder (1964) is universally acknowledged. A woman may have more or fewer children, and she may have them earlier or later in her life. It makes sense to ask for period measures of total fertility which adjust for changes in the timing of childbearing independent of changes in numbers of children at the individual level. John Bongaarts and Griffeth Feeney (1998) provided such a fertility measure which has gained many adherents, including the present author.

In the study of mortality, no distinction between quantum and tempo exists at the individual level. A person has one death, his or her own, and mortality pertains to whether death comes early or late. It makes no obvious sense to adjust away the effects of changes in the timing of death, thus adjusting away changes in mortality itself. New papers by Bongaarts and Feeney (2002) and (in this volume p. 11) came as a surprise, offering a family of measures put forward to adjust period life expectancy for effects which they called tempo distortions. The different measures in the family coincide with each other under a condition on the age and time-specific hazard rates called the "Proportionality Assumption" which the authors find to be approximately satisfied by adult mortality schedules in various developed countries over some recent decades.

Any measure measures something. The question is whether the something being measured is a version of current period life expectancy freed from some kind of distortion. This chapter puts the spotlight on a representation which helps in visualizing what the new measures do measure. The new measures do not measure current mortality conditions but rather the cumulative effects of earlier mortality conditions. The period life expectancy does measure current mortality conditions.

The words "current conditions" are used here in their ordinary English-language sense. Current mortality is the mortality that can be currently observed by counting deaths and counting person-years at risk. An alternative usage introduced by Vaupel (2002) in which "current conditions" is used as shorthand for "current latent conditions" in a latent-structure representation is discussed in Section 6.

The representation of the Bongaarts-Feeney measures takes the form

$$M(t) \approx \int_{-\infty}^{t} w_t(\tau) e_0(\tau)\, d\tau \tag{1}$$

Here $e_0(t)$ is period life expectancy at time t. (In applications, e_0 is replaced by e_{30} since the approach is intended solely for adult mortality.) $M(t)$ is a Bongaarts-Feeney measure of adjusted life expectancy. For each t, $w_t(\tau)$ is a probability distribution defining weights over a set of lagged time periods $\tau < t$. As functions of the lag $s = t - \tau$, the weights are nearly exponential and nearly independent of t.

The representation is an approximation which holds to first order in the time derivative of M under the hypothesis that Bongaarts and Feeney's Proportionality Assumption is sufficiently nearly satisfied that the different measures in the family are equivalent to each other within the limits of the approximation. Details are spelled out in Section 3.

The representation shows that the Bongaarts-Feeney measure M is a weighted average of period e_0 values from the recent past. The period life expectancy itself at time t depends only on current age-specific hazard rates for time t. The Bongaarts-Feeney measure depends on past as well as current age-specific hazard rates. When longevity has been increasing, past values of e_0 are lower than current values, and the Bongaarts-Feeney measure averages over these lower past values and produces a value below present-day e_0. When longevity has been decreasing, past values exceed current values, and the Bongaarts-Feeney measure averages over these higher past values and hovers above present-day e_0.

The word "distortion" is out of place when contrasting M to e_0. The measures measure different things. If one cares about average mortality levels in the recent past, one can use one of the Bongaarts-Feeney measures. If one cares about mortality levels under current conditions, one can use the period life expectancy.

The representation (1) gives concrete form to the general observation that the Bongaarts-Feeney mortality measures are functions not solely of current mortality but also of the population age structure that would be produced by past mortality conditions given a hypothetical constant stream of prior births. This dependence was pointed out in their initial paper (2002 ,p. 23). Bongaarts and Feeney noted that their adjusted measure could not be calculated directly from period hazard rates "because $\mu^*(a,t)$ [their adjusted hazard rates] are in general not observable". They discussed a need for a century or more of age-specific death rates for their calculations.

In this same early paper, Bongaarts and Feeney (2002, Eq. 12), introduced a differential equation (originally under Gompertzian assumptions) which agrees to first order with equation (7) of Section 3. They imposed a boundary condition which allowed them to estimate values of their measure at each time t from the sequence of prior values of period life expectancy, in effect implementing a numerical calculation of the representation (1). The equation for $M(t)$ in terms of coefficient values for time t is a differential equation, not an algebraic equation. It is therefore not a recipe for calculating the value of M at time t solely from period information for time t. The solution M is only defined with respect to the boundary conditions and time trajectories of the coefficients. This dependence on the past is the fundamental property of the Bongaarts-Feeney mortality measures.

Definitions of the measures are given in Section 2. The representation is presented in Section 3 with examples in Section 4 and discussion in Section 5. Proposals to relate the Bongaarts-Feeney measures to latent structure representations of mortality are analyzed in Section 6. Unlike the adjusted life expectancies, Bongaarts and Feeney's adjusted total fertility measure at a time t depends only on age-specific fertility rates in an arbitrarily small neighborhood of t. It is independent of population age structure and independent of past levels of fertility. This fundamental difference between the proposed mortality adjustments and the fertility adjustments precludes any close analogy between them. The difference is highlighted in Section 7, which presents an interpretation of the fertility adjustments in terms of a process of birth-age standardization.

2 Measures

Clarity is promoted by expressing the measures under discussion in standard demographic notation.

$\mu(a,t)$ is the hazard rate at age a at time t;

$N(a,t) = N(0,t)\exp(-\int_0^a \mu(x,t-a+x)\,dx)$ is the number of population members aged a at time t expressed as a density with respect to $da\,dt$;

$N(0,t) \;= 1$ is a normalization on initial cohort size which keeps the number of births per unit time constant at unity;
$e_0(t) \;= \int \exp\left(-\int_0^a \mu(x,t)\,dx\right) da$ is the period expectation of life;
$d(a,t) \;= N(a,t)\mu(a,t)$ is the count of deaths at age a and time t;
$D_+(t) \;= \int N(a,t)\mu(a,t)\,da = \int d(a,t)\,da$ is the period count of total deaths;
$N_+(t) \;= \int N(a,t)\,da$ is the period total population;

The basic condition on the population distribution $N(a,t)$ is the normalization which sets the size of every cohort at birth equal to unity, equivalent to dividing the numbers aged a at time t by the numbers aged 0 at time $t-a$ for all a and t. Given this normalization, the measures $M_1 \dots M_4$ introduced in the notation of their *PNAS* article (Bongaarts and Feeney 2003, also published in this volume p. 11) correspond to familiar population quantities:

- M_1 is the total population count $N_+(t)$, equal to the "Cross-Sectional Average Length of Life" $CAL(t)$ introduced by Nicolas Brouard (1986) and Michel Guillot (2003);
- M_2 is the period mean age at death, $MAD(t)$ in the terminology of Bongaarts and Feeney (in this volume p.29) , given by

$$\int aN(a,t)\mu(a,t)\,da/D_+(t);$$

- M_3 is the period life expectancy $e_0(t)$;
- M_4 is an adjusted life expectancy defined by

$$M_4(t) = \int \exp\left(-\int_0^a \frac{\mu(x,t)}{1-\frac{d}{dt}M_1(t)}\,dx\right) da \tag{2}$$

In Bongaarts and Feeney (in this volume p.29), the derivative of M_1 in (2) is replaced by the derivative of M_2, producing a closely related measure which might reasonably be called M_5.

The total population count changes over time by the addition of births and subtraction of deaths, so the time derivative of $N_+(t) = M_1(t) = CAL(t)$ is $1 - D_+(t)$. Dividing the hazard rates for time t at every age by the count of total deaths, retaining an unchanged population $N(a,t)$ at risk, resets the total deaths to unity. In other words, the rates inside the integral in the definition of M_4 are rates which, given the age structure, would make period deaths equal normalized period births. Caution is advisable in interpreting these measures. The measure CAL does not always correspond to the statistical expectation of a waiting time, even though the formula might seem to suggest so. The measure M_4 employs a proportional adjustment to hazards, whether or not hazards have been changing proportionally in the past.

The "Proportionality Assumption" of Bongaarts and Feeney (in this volume p.11) is a condition on the partial derivatives of $N(a,t)$ for all a and t in terms of a function $r(t)$ varying in a neighborhood of zero:

$$\frac{\partial N(a,t)}{\partial t} = -r(t)\frac{\partial N(a,t)}{\partial a} \quad (3)$$

(This $r(t)$ is the same as $1 - p(t)$ in Bongaarts and Feeney (in this volume p.11, Eq. 6).) It should be borne in mind that the condition expressed in terms of N for given a and t involves a whole family of constraints on the hazard rates μ at earlier ages and earlier times which produce the value of N and its rates of change with age and time. It is not a local condition confined to a neighborhood of a and t.

Equation (3) determines a family of parallel curves giving contours of constant N over time. The shape of the age distribution is preserved and shifted up or down as shown in Bongaarts and Feeney (in this volume p.11). Specifically, setting $F(t) = \int_0^t r(\tau)\, d\tau$, (3) provides for a vanishing time derivative for $N(a + F(t), t)$, allowing $N(a,t)$ to be expressed in terms of $N(a,0)$. The hazards $\mu(a,t)$, defined from the partial derivatives of the logarithm of N at time t and hence from the partial derivatives at time zero, have to take the form

$$\mu(a,t) = (1 - F'(t))\psi(a - F(t)) \quad (4)$$

Here ψ is a non-negative function of age a vanishing for negative a, defined from derivatives of the logarithm of N at time zero.

Three other results proved in Bongaarts and Feeney (in this volume p.11) follow readily from (3). Integrating both sides of (3) with respect to a shows that the time derivative of $M_1(t)$, that is, of $CAL(t)$, is given by $M_1'(t) = 1 - D_+(t) = r(t)$. Integrating $\int ad(a,t)da = \int aN(a,t)\mu(a,t)da$ by parts yields the equality $M_2 = M_1$. Writing the hazard rate quotient $\mu(a,t)/(1 - r(t))$ as the partial derivative with respect to a of $-\log(N(a,t))$ shows that $M_4 = M_1$.

3 Representation of M

When the Proportionality Assumption holds, the equality of M_1, M_2, and M_4 allows us to set $M = M_1 = M_4$ in the equation defining M_4 and obtain a differential equation satisfied by the common values of M_1, M_2, and M_4:

$$M(t) = \int \exp\left(-\int_0^a \frac{\mu(x,t)}{1 - \frac{d}{dt}M(t)}\, dx\right) da \quad (5)$$

When the Proportionality Assumption does not hold exactly, this equation can also be regarded as defining a measure of interest in its own right, which could take a place beside M_1, M_2, and M_4 in the family of measures. Indeed, the original measure introduced in (2002, 23) was a solution to a version of this equation. It is expected that all these measures will be close to each other when the Proportionality Assumption is approximately valid. One could, for example, stipulate that $\mu(a,t)$ agree to first order in some parameter ϵ with the corresponding values for a set of hazard rates that do satisfy the Proportionality Assumption. Weaker conditions might also suffice to guarantee agreement to order $O(\epsilon)$ among the measures. All that is at stake here is approximate consistency among the different choices of measures in the family. Once Equation (5) is in hand, the further arguments leading to our representation do not depend on the Proportionality Assumption.

We obtain our representation by expanding the right-hand side of (5) in powers of $r = M'(t)$ for each t. The value of the right-hand side at $r = 0$ is the period life expectancy. The inner integrand $\mu/(1-r)$ in (5), being proportional to μ, brings into play the familiar machinery of proportional hazards. As in (1985, 80), the derivative with respect to r is a multiple of "lifetable entropy" given, at $r = 0$, by minus the quantity

$$g(t) = \int_0^\infty e^{-\int_0^a \mu(x,t)\,dx} \int_0^a \mu(y,t)\,dy\,da \tag{6}$$

The result is an equation which is a first-order approximation to (5) when $M'(t)$ is uniformly small:

$$M(t) = e_0(t) - g(t)M'(t) \tag{7}$$

Under appropriate regularity conditions mentioned below, the differential equation (7) has a unique solution bounded at minus infinity given by the integral already presented in Equation (1):

$$M(t) = \int_{-\infty}^t w_t(\tau) e_0(\tau)\,d\tau \tag{8}$$

The time-dependent weights $w_t(\tau)$ are given in terms of the reciprocals of $g(\tau)$ by the expression

$$w_t(\tau) = g^{-1}(\tau)\exp\bigl(-\int_\tau^t g^{-1}(s)\,ds\bigr) \tag{9}$$

For each t, these positive weights integrate up to unity over τ and define a probability distribution. The inner integral in (9) can be used to define an alternative time-like coordinate in terms of which the weights become exponential functions.

It is easy to verify that (1) formally satisfies (7) by differentiating the right-hand side of (1) with respect to the argument t which occurs both in the limit of integration and in the function $w_t(\tau)$. The derivative of $w_t(\tau)$ with respect to t is $-w_t(\tau)/g(t)$ and $w_t(t) = 1/g(t)$.

$$\begin{aligned} e_0(t) - g(t)M'(t) &= e_0(t) - g(t)\frac{d}{dt}\int_{-\infty}^{t} w_t(\tau)e_0(\tau)\,d\tau \\ &= e_0(t) - g(t)w_t(t)e_0(t) - g(t)\int_{-\infty}^{t}\frac{d}{dt}w_t(\tau)e_0(\tau)\,d\tau \\ &= e_0(t) - g(t)(1/g(t))e_0(t) - \int_{-\infty}^{t} -w_t(\tau)e_0(\tau)\,d\tau \\ &= M(t) \end{aligned}$$

The function $g(t)$ is strictly greater than zero, so long as lifetable deaths in the period lifetable are not concentrated all at a single age, which is always true if μ is finite. We assume further that $1/g(t)$ and $e_0(t)/g(t)$ are integrable on bounded intervals and that $g(t)$ is bounded, making the weights in (9) finite and the solution in (8) the unique one bounded at minus infinity (1955, pp. 67,97).

In expanding the right-hand side of (5), we could have expressed the difference between the values at zero and at r using the derivative evaluated at r instead of at zero. The answers would agree to first order. The derivative at zero from (6) has the advantage of being a purely period measure. But the derivative at r, obtained from (6) by substituting $\mu/(1-r)$ for μ, is also informative. It is exactly constant when the Proportionality Assumption is exactly valid. It follows that $g(t)$ must be nearly constant so long as the Proportionality Assumption is nearly valid, making the weights $w_t(t-s)$ as a function of the lag s nearly equal to a fixed exponential distribution $(1/g)\exp(-s/g)$.

A clear conclusion follows from this representation: This candidate for a "tempo-adjusted expectation of life" is, to first order, an explicit moving average of recent past values of the period expectation of life. When levels of survival are increasing, current values of $e_0(t)$ exceed past values. What Bongaarts and Feeney are interpreting as a "tempo distortion" is simply the difference produced by focussing on the present instead of focussing on the recent past.

Period life expectancy is sensitive to sudden changes affecting mortality at many ages. It is meant to be so. That is an advantage, not a drawback. When period life expectancy falls, deaths are surging. People are dying. It is no mirage or distortion of reality.

A rise or fall in hazard rates concentrated in time but spread over many ages will have effects spread over many cohorts, so a large temporary change in period life expectancy should and does correspond to a suite of small changes in cohort life expectancy for many cohorts. Averaging period measures over a stretch of time that includes large parts of the lifespans of many cohorts naturally leads to values in line with the average values of the corresponding cohort measures. The retrospective averaging implemented by the Bongaarts-Feeney measures has this kind of outcome. The period life expectancy, for its part, is a faithful indicator of current conditions.

4 The moving average

To see how the representation of the Bongaarts-Feeney measures works out in practice, consider Swedish female adult mortality, example B of Bongaarts and Feeney (in this volume p.11, Figure 6). The measures are only meant to apply after about age 30, so we let age $a = 0$ correspond to age 30 and condition on survival to that age. Single-year age-specific mortality rates from 1861 to 2001 are taken from the Human Mortality Database (2004) assembled by John Wilmoth at Berkeley, allowing calculation of CAL and MAD for ages above 30 from 1941 onwards.

In these Swedish data, the entropy measure g (for ages above 30) is close to 9 back to about 1945, a level reached after a gradual long-term drop from Nineteenth Century values around 13. The gradual changes in g imply slight changes in exponential weights, but for measures after 1941 the moving average (8) with changing weights (9) is only slightly different from a moving average with fixed exponential weights set with $g = 9$. (The mean difference is 0.063 years and the maximum difference is 0.186 years.) Thus we are essentially dealing with a simple exponential distribution with a nine-year mean. The Bongaarts-Feeney measures CAL, MAD, and M_4, where they agree with each other, are given by a simple exponential weighted average of past values of period life expectancy, with an average look-back time of 9 years.

For example, consider the calculation of M for $t = 2001.0$. The year from December 2000 back to January 2000 is the first year back. The weight for this year, applied to period life expectancy centered at mid-year, is the integral of $(1/9)\exp(-s/9)$ between 0 and 1, or $e^{-0/9} - e^{-1/9}$. The weight for the second year back (1999) is $e^{-1/9} - e^{-2/9}$, etc. M is the weighted average, the sum of weights times life expectancies back over time:

$$\begin{aligned} M &= (e^{-0/9} - e^{-1/9})e_{30}(2000) + (e^{-1/9} - e^{-2/9})e_{30}(1999)\ldots \\ &= (0.10516)(52.587) + (0.09410)(52.451)\ldots \\ &= 51.55 \end{aligned}$$

For 2001, comparing M to values of CAL, MAD, and M_4 calculated directly from single-year mortality rates, we see that the weighted average $M = 51.55$ years falls a little above $CAL = 51.43$ years between $M_4 = 51.52$ years and $MAD = 51.58$ years. The period life expectancy e_{30} is a year higher, at 52.63 years.

It is instructive to see with formulas how the weighted average recovers the values of CAL and MAD when the Proportionality Assumption holds. As before, we let $a = 0$ correspond to human age 30. Thanks to (4), we have $\mu(a,t) = (1 - F'(t))\psi(a - F(t))$ with a baseline age schedule ψ and a shift function $F(t)$ whose time derivative equals the proportionality factor $r(t)$. Values of CAL and MAD at time zero are given by $\eta = \int exp(-\int_0^a \psi(x)\,dx)\,da$ and the values at time t include the shift $F(t)$:

$$CAL(t) = MAD(t) = \eta + F(t) \tag{10}$$

The same Taylor expansion as in (7) for life expectancies under proportional hazards yields

$$e_0(t) \approx \eta + F(t) + gF'(t) \tag{11}$$

Here the coefficient g can be set equal to the rescaled entropy derived from ψ which is constant over time. It is given by formula (6) with $\psi(x - F(t))$ in place of $\mu(x, t)$. Since ψ vanishes for negative a and the outer integral runs over all a, the formula is unchanged when $F(t)$ is deleted from the arguments of ψ, leaving an expression independent of t.

The weights are given by $w_t(t - s) = (1/g)\exp(-s/g)$. The weighted average is an integral with respect to this exponential probability distribution whose mean is g:

$$\begin{aligned} M &= \int_0^\infty e_0(t-s)(1/g)e^{-s/g}\,ds \\ &= \int (\eta + F(t-s) + gF'(t-s))\,(1/g)e^{-s/g}\,ds \\ &= \eta + F(t) - \int (F(t) - F(t-s))(1/g)e^{-s/g}\,ds \\ &\qquad\qquad + \int F'(t-s))\,e^{-s/g}\,ds \end{aligned}$$

Integrating the third term by parts yields $-\int F'(t-s)e^{-s/g}\,ds$, exactly cancelling the fourth term, so that

$$M = \eta + F(t) = CAL(t) = MAD(t) \tag{12}$$

When the proportionality factor $r(t) = F'(t)$ is constant, we have the case of linear shifts analyzed by Goldstein (in this volume) and by Rodriguez (in this volume). The graphs of $e_0(t)$ and $CAL(t) = MAD(t)$ are parallel straight lines with slope r. Lagged life expectancy is the linear function $e_0(t - s) = e_0(0) + r(t - s)$. Its average is $e_0(0) + r(t - g)$ since the average value for s is g. Thus $CAL(t)$ comes out to be the lagged value $e_0(t - g)$.

When g is calculated from a hazard function given by a Gompertz model $\alpha e^{\beta a}$, we have $g = (1/\beta) - (\alpha/\beta)e_0$. The second term is usually two orders of magnitude smaller than the first term, so $g \approx 1/\beta$. Suppose that hazards change over time according to a Gompertz model with constant β and more or less exponentially declining $\alpha(t)$ approximated, say, by $\alpha(0)\exp(-r\beta t)$. Suppose also that $\alpha(0)$ is small enough that young mortality can be neglected or set to zero. Then the Proportionality Assumption comes to be satisfied with something close to a linear shift of slope r. In principle the Proportionality Assumption could hold under different, non-Gompertzian conditions, but in the empirical examples known to the present author it seems to arise in this way.

Since the weights in the moving average representation fall off exponentially, the remote past has negligible impact, and the full moving average can be replaced by an average reaching back over a finite span of years. The representation is meant to hold to first order in M'. In the Swedish data, M' is on the order of 0.15 and second-order terms are on the order of 0.02. A span of $6g$ years, or 54 years, includes all but $\exp(-6g/g) = \exp(-6) = 0.002$ of the weight from the exponential distribution. Periods that represent the early adult life experience of cohorts older than $30 + 54 = 84$ years have only minor impact on CAL and MAD.

Mathematically speaking, when the Proportionality Assumption is only tenable for some limited span $t > T$, the solution (8) to the differential equation (7) (which is the solution vanishing at minus infinity) needs to be replaced by the solution satisfying an appropriate boundary condition at $t = T$, that is, one making $M(T) = CAL(T)$. The moving average only reaches back to T and the term introduced by the boundary condition tapers exponentially as time goes by.

Figure 1 shows mortality measures for Swedish women from 1941 to 2001, all calculated beyond age 30. The upper solid line is period life expectancy. The lower solid line is CAL, trending steadily upward with an average slope of 0.17 per year. The dashed line for MAD hugs CAL from 2001 back to 1975 but separates from it at earlier times just outside the range of years shown in Bongaarts and Feeney (in this volume p.11, Figure 6B). The separation signals failure of the Proportionality Assumption. The moving average M is the dotted line. The measure M_4, not shown in the plot, is close to M before 1970 and close to CAL after 1980. Where CAL and MAD diverge from each other, the moving average M turns out to strike a balance between them.

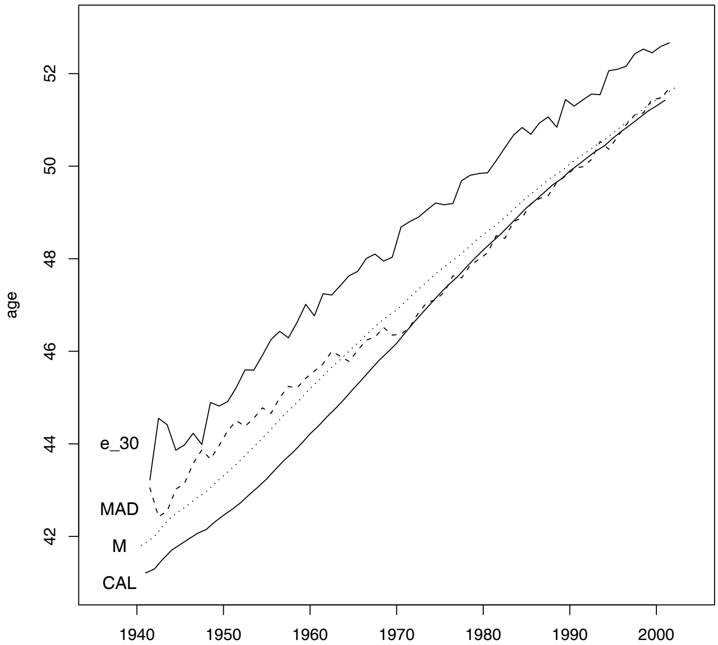

Fig. 1. Mortality measures for Swedish women 1941-2001.

5 Period counts of deaths

Period counts of deaths play an important role in the formulas for the mortality measures and an important role in the analogies which Bongaarts and Feeney (in this volume p.29) seek to develop. In their papers they give a new name to the period count of deaths $D_+(t)$, calling it the "Total Mortality Rate" or "*TMR*". They liken this quantity to the Total Fertility Rate, Total First Marriage Rate, and other indices for processes that, unlike mortality, admit a distinction between quantum and tempo at the individual level.

Ordinarily, one would expect instead to define the "*TMR*" with a formula parallel to the formula for the *TFR*:

$$TFR(t) = \int f(a,t)\,da \tag{13}$$

$$TMR(t) = \int \mu(a,t)\,da \tag{14}$$

The period count of deaths is a count, not a rate. Bongaarts and Feeney defend their practice of calling it a rate by taking the usual denominator, those at risk of the event, and adding on a set of "ghosts", those who would have been at risk had they not exited from the population by dying. The same construction can be applied with fertility to obtain period counts of births $B_+(t)$ from the fertility rates, albeit counts that need not agree with initial cohort sizes:

$$B_+(t) = \int N(a,t)f(a,t)/N(0,t-a)\,da \tag{15}$$

$$D_+(t) = \int N(a,t)\mu(a,t)/N(0,t-a)\,da \tag{16}$$

The tempo adjustment for fertility in (1998) is an adjustment to the *TFR*, not $B_+(t)$, whereas the tempo adjustments for mortality in Bongaarts and Feeney (in this volume p.11) involve adjustments to $D_+(t)$, not to the *TMR*, which is generally infinite.

The normalization which enforces a constant unit stream of births into the population means that the population is increasing when and only when $D_+(t)$ is less than 1, that is, when births exceed deaths, and decreasing when $D_+(t) > 1$. This quantity $D_+(t)$, the period count of deaths per unit birth, is less than 1 if mortality has been higher in the past than in the present. The higher death rates of the past deplete the surviving population at risk of dying and thus reduce current deaths. This outcome is not a tempo effect. It can remain true even if current mortality is increasing rather than declining.

Replacement of the hazard rates $\mu(a,t)$ by rates $\mu(a,t)/D_+(t)$ in the formula for M_4 does, as mentioned, bring total deaths into equality with normalized total births so long as the population age structure is retained unaltered. However, this transformation cannot be achieved by a systematic reassignment of times of death, because any reassignment necessarily alters the population age structure. The substitution underlying the M_4 measure is a form of standardization for the total flow of deaths which is difficult to interpret in terms of any assumptions about individual experience.

6 Current latent conditions

A question arises as to whether measures equivalent or similar to those of Bongaarts and Feeney might be definable from some latent structure representation of mortality. Vaupel (in this volume p.93) writes about such possibilities. An example predicated on the heterogeneous frailty model of Vaupel, Manton, and Stallard (1979) is given by Vaupel (2002). Starting from any $\mu(x,t)$, for each choice of a frailty dispersion parameter σ, one can define hypothetical latent baseline hazards $\mu^o(x,t)$ by the equation

$$\mu^o(x,t) = \mu(x,t)\exp\bigl(\sigma^2 \int_0^x \mu(a, t-x+a)\, da\bigr) \tag{17}$$

This formula is a *representation.* For any observed $\mu(x,t)$ it supplies a latent $\mu^o(x,t)$ which will reproduce it. From μ^o, Vaupel defines a measure which he calls a version of life-expectancy "under current conditions", that is, under current latent rather than current observed conditions.

Vaupel's frailty-based measures are well defined but they are at a far remove from the Bongaarts-Feeney measures. They depend on population heterogeneity, whereas Bongaarts and Feeney's arguments apply to wholly homogeneous populations. In empirical cases like the Swedish series, the frailty-based measures fluctuate in tandem with period life expectancy, lack the smoothing properties of CAL, MAD, and M_4, and differ only by small amounts from period life expectancy.

The interesting feature of the frailty-based measures is conceptual. Although current μ^o is calculated from past values of μ, one can imagine an experiment for measuring current μ^o from current observations. Take a random sample of people who had lived in a country with negligible mortality up to age x, transplant them to a country beset by μ, and identify μ^o with any higher hazards that such higher-mean-frailty refugees experience. In practice, debilitation probably dominates culling, and the experiment would founder, but the concept is coherent.

Recognizing the absence of connection between his frailty-based measures and the actual Bongaarts-Feeney measures, Vaupel (in this volume p.93) goes on to sketch a different approach which might also come under the heading of "mortality under current latent conditions". The latent variables are tickets associated with predestined ages of death. Life is like a pastiche of an old Beatles song

"I have a ticket to die."

Vaupel's chapter presents examples rather than a general treatment. In some examples, the proposal is to have ticket values that can change either deterministically or stochastically over time, depending on the current ticket value but not on the current age of the holder. When a person's age catches up with his or her current ticket value, the person dies.

We may write $V(U,t)$ for a ticket process started at an initial state indexed by U and varying over time t. U has some probability distribution across the population. In versions with deterministic transitions, $V(U,t)$ is a function of U and t, usually a continuous function. In versions with stochastic transitions, $V(U,t)$ is a Markov process started at a state indexed by U unfolding either with discrete time steps and discrete states corresponding to age groups, or with continuous time and age. The distribution of ticket values at birth for a cohort born at time τ is the marginal distribution of $V(U,\tau)$ generated by the randomness in U and the randomness, if any, in V given U. The distribution of ages at death for the cohort is the distribution of the random variable

$$\min\{x : V(U, \tau + x) \leq x\} \tag{18}$$

A person dies when he or she first reaches an age coinciding with the age currently on his or her ticket.

Detailed treatment is beyond the scope of this chapter, but we proffer some reflections based on early analysis.

If $V(U, t)$ can be specified, then a current measure can be defined to equal the period mean of V. That part is easy. What is difficult is the representation problem. No equation like (17) is at hand for taking observed $\mu(x, t)$ and writing down some specific V that generates it. Without a representation formula, one has no well-defined measure and nothing to compare with Bongaarts and Feeney's proposed adjustments.

One can, of course, make up ticket models *de novo* and endeavor to test their goodness of fit to μ values like the Swedish series. That may be interesting, but testing goodness of fit is not what Bongaarts and Feeney are doing. They are defining measures. From any μ, they obtain measures to contrast with period life expectancy, and they argue for an automatic adjustment to period life expectancy whenever observed past hazards differ from present ones.

To make ticket models relevant to Bongaarts and Feeney's proposals, one needs, then, to focus on the representation problem. With deterministic transitions, the only apparent prospect is a version of Feeney's (in this volume) derivations. See also Wilmoth (2005). We can let U be a uniform random variable marking a cohort member's predestined proportional placement in a rank ordering of the cohort from oldest to youngest by age at death. Define the quantile function

$$Q(U, \tau) = \min\{x : \int_0^x \mu(a, \tau + a)\, da = -\log(U)\} \tag{19}$$

For each fixed U and t , the equation $Q(U, t - v) = v$ may have a unique solution v, and if it does, we can set $V(U, t) = v$. In such cases the measure, the period mean of V, comes out to equal CAL.

However, unique solutions do not always exist. The same cases that defeat Feeney's (in this volume) attempt at generality prevent this construction from yielding a general representation of mortality schedules. Cases that fail occur when the partial derivative of Q with respect to τ takes values less than or equal to -1. These tickets are intrinsically cohort objects that resist alignment by periods. A person's U value is a cohort percentage. Today's ticket values only have meaning insofar as we match values for current survivors to values for current decedents who share the same U, fixed by their cohort's prior history. Unlike Vaupel's frailty-based μ^o values, the current values of these latent variables have no independent reality in the present that can be easily discerned. No experiment is on the table which would allow us to elicit present-day ticket values from present-day observations alone.

Turning to ticket models with stochastic transitions, we encounter the representation problem in a different guise. Here the specification of $V(U,t)$ is drastically underdetermined. Analysis in continuous time is technically challenging, but the issues can be scrutinized in discrete time with Markov chains with finitely many states corresponding to age groups numbered from 1 to k. Each transition matrix at each time t contains $k(k-1)$ elements that need to be determined. The observed distribution of deaths for each cohort, which the model has to match, is specified by $k-1$ quantities. Thus, ignoring endpoint effects, T cohorts give $(k-1)T$ equations in $k(k-1)T$ unknowns. Already with $k=3$, a wide range of different solutions are allowed. Subject to some messy inequalities, one can choose one's solution at will to make the resulting period measure agree with any of a wide variety of arbitrary sequences. Without some natural set of identifying restrictions, as yet to be discovered, the ticket model framework with stochastic transitions gives nothing definite to compare with Bongaarts and Feeney's measures.

7 Total fertility

It would be an unhappy outcome if the limitations of the proposed measures for adjusted life expectancies undermined confidence in the tempo-adjusted measures for total fertility proposed earlier by Bongaarts and Feeney (1998). Unlike the mortality measures, the fertility measures are standardized indicators of current conditions. The adjusted total fertility rate at time t depends only on age-specific fertility rates $f(a,t)$ in an arbitrarily small neighborhood of t. It does not depend on age structure and it does not depend on past fertility rates. It has a direct interpretation in terms of individual experience. This section offers a formulation of the adjusted fertility measures which highlights these attractive features.

Age-specific fertility rates $f(a,t)$ are written here as a function of continuous age a and continuous time t. As usual, the period Total Fertility Rate $TFR(t)$ and period mean age at childbearing $A(t)$ are given by

$$TFR(t) = \int f(a,t)\, da \tag{20}$$

and

$$A(t) = \frac{\int a\ f(a,t)\, da}{TFR(t)} \tag{21}$$

A simple procedure for producing an adjusted index is to define a coordinate transformation which, in effect, reassigns the timing of births within cohorts leaving numbers of births invariant within cohorts. The transformation is chosen so that, after reassignment has been carried out, a period computation of mean age at childbearing would give a constant outcome, thus erasing period variations in timing. The post-reassignment value for the mean age can be set arbitrarily to some standard value A_s, perhaps most sensibly to a long-term average for cohort mean ages at childbearing conditional on survival through childbearing years.

The transformation Ψ is given by

$$a \rightarrow \alpha = a - A(t) + A_s \tag{22}$$

$$t \rightarrow \tau = t - A(t) + A_s \tag{23}$$

We assume that $A(t)$ is differentiable and we impose the reasonable assumption that the period mean age at childbearing never increases by as much as a full year per year, so that the time derivative $A'(t)$ is always less than 1. Then the transformation is invertible and has a finite Jacobian given by

$$\frac{\partial\, \alpha, \tau}{\partial\, a, t} = 1 - A'(t) \tag{24}$$

The inverse function $t(\alpha, \tau)$ only depends on τ. Age-specific fertility rates after reassignment are given by

$$\tilde{f}(\alpha, \tau) = \frac{f(a(\alpha, \tau), t(\tau))}{1 - A'(t(\tau))} \tag{25}$$

This definition guarantees agreement between integrals over subsets S in the Lexis plane:

$$\int\int_S \tilde{f}\, d\alpha\, d\tau = \int\int_{\Psi^{-1}S} f\, da\, dt \tag{26}$$

An adjusted or standardized Total Fertility Rate *STFR* can be defined from $\tilde{f}$:

$$STFR(\tau) = \int \tilde{f}(\alpha, \tau)\, d\alpha = \frac{TFR(t(\tau))}{(1 - A'(t(\tau))} \tag{27}$$

These integrals are taken over α for fixed τ, unlike the double integrals of Equation (26). It is readily verified that the period mean age of childbearing defined from $\tilde{f}$ remains constant at a level A_s and that integrals of $\tilde{f}$ along diagonals of the Lexis diagram are identical to integrals of f itself.

Kohler and Philipov (2001) introduce this Jacobian-based formulation for tempo adjustments, although they deviate from it in the definition of their own generalized measure. The transformation shifts fertility backwards or forwards along cohort lifelines on the Lexis diagram. The cohort quantum of fertility measured by a cohort *TFR* (conditional on survival) is unchanged. The positioning of births along the lifelines of mothers in the cohort is adjusted in such a way as to hold the transformed period mean age at birth constant at the chosen standard value A_s.

The size of *STFR* defined by Equation (27) is the same as Bongaarts and Feeney's tempo-adjusted *TFR*. It is expressed as a function of the hypothetical coordinate τ rather than the real time coordinate t, but, if desired, it can be attributed back to t, since the transformation is invertible. Although τ depends on the choice of the standard age A_s, the measure itself does not depend on it. The mathematics would be the same if we took A_s equal to zero, but visualization is easier if we take it equal to some realistic benchmark age.

The reassignment process expressed by our coordinate transformation can be regarded as a kind of standardization. It differs from familiar kinds of demographic standardization like the standardization of Crude Birth Rates for effects of age distributions. But it serves a parallel purpose. Just as one asks, "What would a Crude Birth Rate turn into if population age group sizes were set to standard values?", one can ask, "What would a Total Fertility Rate turn into, if period mean ages at childbirth were set to a standard values?" In this sense, the Bongaarts-Feeney tempo adjustment for fertility can be viewed as a process of birth-age standardization.

This way of viewing the measure clarifies several issues. Bongaarts and Feeney's fertility measure does not depend on any behavioral assumptions about fertility, any more than an age-standardized birth rate depends on behavioral assumptions. It does, however, suggest a thought experiment, because one can imagine individuals changing the timing of their births in such a way as to change the observed *TFR* into the adjusted or standardized one.

For applications of their measure, Bongaarts and Feeney recommend applying their adjustment separately parity-by-parity to birth-order-specific frequencies. These are not the same as age and parity-specific rates. Each numerator includes only births of a given parity while the corresponding denominator includes person-years from women of all parities. These quantities sum up to the overall age-specific fertility rates, so they comprise an additive decomposition. Conceptual difficulties arising from reliance on such frequencies or "rates of the second kind" in place of occurrence-exposure rates or "rates of the first kind" have been pointed out by Van Imhoff and Keilman (2000).

As a formal procedure, nothing prevents the kind of standardization achieved by Equation (22) from being applied separately to any additive decomposition of age-specific fertility rates:

$$f(a,t) = \sum_i f_i(a,t) \tag{28}$$

Any such decomposition in terms of some categorization of births can be accommodated. Birth order is one option, but mother's marital status, mother's education, region of birth, and sex of baby are among a host of others. When a transformation is applied to each f_i and the resulting $STFR_i$ are added together to produce an aggregate $STFR$, the result is an index which has been standardized for changes in period mean ages at childbearing within each of the subgroups. No behavioral claims need be at issue. It is probably a mistake to make a fetish of the decomposition by parity. The fact that one particular breakdown among many would allow a complicated re-expression in terms of occurrence-exposure rates need have no deep bearing on the nature of the adjustment.

In summary, Bongaarts and Feeney's tempo adjustment for the Total Fertility Rate can be viewed as a process of standardization. It erases effects of changes in period mean ages while preserving cohort quantum (conditional on survival). There is a clear distinction at the individual level between something that is being reset and something that is being left invariant. The adjustment does not rely on any behavioral model or structural representation of fertility processes. Like traditional standardized measures, it is a valuable device for comparing cases, controlling for a particular source of variation.

No such process of standardization makes sense in the context of mortality, because there is no distinction at the individual level between something to reset and something to leave invariant. The timing of a person's death is what is being assessed when we assess mortality. Controlling for changes in the timing of death is tantamount to controlling for mortality itself.

Discussions of quasi-behavioral models and structural representations in the context of Bongaarts and Feeney's proposed mortality measures serve to highlight the gulf between these measures and their fertility measure. No elaborate modeling is required with fertility.

Bongaarts and Feeney's adjusted Total Fertility Rate is a current measure, whose value at a time t depends only on values and slopes of age-specific fertility rates at time t. Altogether otherwise, the mortality measures they propose as alternatives to period life expectancy are not current measures. They average over mortality conditions observed in the past. Under the Proportionality Assumption which makes the measures coincide with each other, the measures average over conditions in the past in a particular simple way, as a weighted moving average of prior period life expectancies, as shown in this chapter.

Mortality measures like CAL and MAD are valuable for studying changing hazard schedules, smoothing as they do over sudden changes. Everyone agrees that changing hazards make cohort life expectancies diverge from period life expectancies and that the divergence is worthy of attention. But measures that depend on past hazards serve different purposes from period life expectancy, which depends on current hazards. The past may differ from the present. This fact is not a "tempo" distortion. Adjustments for "tempo" are only meaningful when there is a meaningful distinction between quantum and tempo in individual experience.

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