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ON MIXTURE FAILURE RATES ORDERING

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ABSTRACT. Mixtures of increasing failure rate distributions (IFR) can decrease at least in some intervals of time. Usually this property can be observed asymptotically as $t \to \infty$. This is due to the fact that the mixture failure rate is 'bent down' compared with the corresponding unconditional expectation of the baseline failure rate, which was proved previously for some specific cases. We generalize this result and discuss the "weakest populations are dying first" property, which leads to the change in the failure rate shape. We also consider the problem of mixture failure rate ordering for the ordered mixing distributions. Two types of stochastic ordering are analyzed: ordering in the likelihood ratio sense and ordering in variances when the means are equal.

Keywords: mixture of distributions, decreasing failure rate, increasing failure rate, stochastic ordering, ordering in the likelihood ratio sense.

1. INTRODUCTION

It is well known that mixtures of decreasing failure rate (DFR) distributions are always DFR. On the contrary, mixtures of increasing failure rate distributions (IFR) can decrease at least in some intervals of time (Block *et al*, 2003). As IFR distributions usually model lifetimes governed by aging processes, it means that the operation of mixing can change the pattern of aging, e.g., from positive aging (IFR) to the negative aging (DFR). It can also slow down the observed via the failure rate process of aging. These facts should be taken into account in applications. It is also clear that those are statistical artifacts and the individual aging is not affected by the operation of mixing.

One can hardly find homogeneous populations in real life and mixtures of distributions usually present an effective tool for modeling heterogeneity. A natural approach for this modeling exploits a notion of a random unobserved parameter (frailty) Z introduced by Vaupel *et al* (1979) in the demographic context. This, in fact, leads to considering a random failure rate $\lambda(t,Z)$. As the failure rate is a conditional characteristic, the 'ordinary' expectation $E[\lambda(t,Z)]$ with respect to Z does not define a mixture failure rate $\lambda_m(t)$ and the proper conditioning should be performed.

A perfect experiment, showing the deceleration in the observed failure rate is performed by nature. It is well-known that the mortality rate of humans obey the Gompertz lifetime distribution (Gompetz, 1825) with exponentially increasing failure rate (mortality rate). Assuming the proportional gamma-frailty model, which describes the heterogeneity of human population:

$$\lambda(t,Z) = Z\alpha \exp\{\beta t\}, \tag{1}$$

where α and β are positive constants, it can be shown that the mixture failure rate $\lambda_m(t)$ is increasing in $[0,\infty)$ and asymptotically tends to a constant as $t \to \infty$. This fact explains recently observed deceleration of human mortality for oldest old (human mortality plateau, as in Thatcher (1999).

In Sections 2 and 3 some supplementary results are stated. In Section 4 we prove the bending down property (Finkelstein, 2005). The steps of this proof are essential for the rest of the paper. While considering heterogeneous populations in different environments the problem of ordering mixture failure rates for stochastically ordered mixing random variables arises. In section 5 we show that the natural type of ordering for mixing models under consideration is ordering in a sense of likelihood ratio (Ross, 1996; Shaked and Shanthikumar, 1993). Specifically, when two frailties are ordered in this way, the corresponding mixture failure rates are naturally ordered as functions of time in $[0,\infty)$. Some specific results for the case of frailties with equal means and different variances are also obtained.

As usually, by terms "increasing" or "decreasing" we mean "non-decreasing" and "non-increasing, respectively.

2. SOME DEFINITIONS

Let $T \geq 0$ be a lifetime random variable with the Cdf $F(t)$ ($\bar{F}(t) \equiv 1 - F(t)$). Assume that $F(t)$ is indexed by a random variable Z in the following sense:

$$P(T \leq t \mid Z = z) \equiv P(T \leq t \mid z) = F(t,z)$$

and that the pdf $f(t,z)$ exists. Then the corresponding failure rate $\lambda(t,z)$ is $f(t,z)/\bar{F}(t,z)$. Let Z be interpreted as a non-negative random variable with support in $[a,b], a \geq 0, b \leq \infty$ and the pdf $\pi(z)$. Thus, a mixture Cdf is defined by

$$F_m(t) = \int_a^b F(t,z)\pi(z)dz .$$

As the failure rate is a conditional characteristic, the mixture failure rate $\lambda_m(t)$ should be defined in the following way (see, e.g., Finkelstein and Esaulova, 2001):

$$\lambda_m(t) = \frac{\int_a^b f(t,z)\pi(z)dz}{\int_a^b \bar{F}(t,z)\pi(z)dz} = \int_a^b \lambda(t,z)\pi(z \mid t)dz, \tag{2}$$

where the conditional pdf (on condition that $T > t$) is:

$$\pi(z \mid t) \equiv \pi(z \mid T > t) = \pi(z)\frac{\bar{F}(t,z)}{\int_a^b \bar{F}(t,z)\pi(z)dz} \tag{3}$$

Therefore, this pdf defines a conditional random variable $Z \mid t$, $Z \mid 0 \equiv Z$ with the same support. On the other hand, consider the following **unconditional characteristic**

$$\lambda_P(t) = \int_a^b \lambda(t,z)\pi(z)dz\,, \tag{4}$$

which, in fact, defines an expected value (as a function of t) for a specific stochastic process $\lambda(t,Z)$. It follows from definition (2) that $\lambda_m(0) = \lambda_P(0)$. The function $\lambda_P(t)$ is a supplementary one, but as a trend function of a stochastic process, it captures the monotonicity pattern of the family $\lambda(t,z)$. Therefore, $\lambda_P(t)$ under certain conditions has a similar to individual $\lambda(t,z)$ shape: if, e.g., $\lambda(t,z), z \in [a,b]$ is increasing in t, then $\lambda_P(t)$ is increasing as well. On the contrary, $\lambda_m(t)$ can have a different pattern: it can ultimately decrease, for instance, or preserve the increasing in t property. However, it will be proved in Section 4, that

$$\lambda_m(t) < \lambda_P(t), t > 0 \tag{5}$$

and under an additional assumptions, that

$$(\lambda_P(t) - \lambda_m(t)) \uparrow,\ t \geq 0\,. \tag{6}$$

Definition 1 (Finkelstein, 2005). *Relation (5) defines the **weak bending down property** for the mixture failure rate, whereas relation (6) is the definition of the **strong bending down property.***

Sometimes the following property can be also of interest

$$\frac{\lambda_P(t)}{\lambda_m(t)} \uparrow, t \geq 0\,. \tag{7}$$

In this paper we shall mostly focus on (5) and (6).

3. MULTIPLICATIVE MODEL

Consider the following specific multiplicative model

$$\lambda(t,z) = z\,\lambda(t)\,, \tag{8}$$

where, $\lambda(t)$ is a baseline failure rate. This setting defines the widely used in applications frailty (multiplicative) model. Equation (1), e.g., is a specific case of this model. Applying relation (2) gives:

$$\lambda_m(t) = \int_a^b \lambda(t,z)\pi(z \mid t)d\theta = \lambda(t)E[Z \mid t]\,. \qquad (9)$$

A conditional expectation $E[Z \mid t]$ ($E[Z \mid 0] \equiv E[Z]$) plays a crucial role in defining the shape of the mixture failure rate $\lambda_m(t)$ in this model. The following result was proved in Finkelstein and Esaulova (2001):

$$E_t'[Z \mid t] = -\lambda(t)Var(Z \mid t) < 0,$$

which means that the conditional expectation of Z is a decreasing function of $t \in [0,\infty)$. On the other hand, (4) turns to

$$\lambda_P(t) = \int_a^b \lambda(t,z)\pi(z)dz = \lambda(t)E[Z \mid 0]\,. \qquad (10)$$

Therefore

$$\lambda_P(t) - \lambda_m(t) = \lambda(t)(E[Z \mid 0] - E[Z \mid t]) > 0 \qquad (11)$$

and relation (5) holds, whereas under the additional sufficient condition that $\lambda(t)$ is increasing, the bending down property in a strong sense (6) takes place. The function

$$\frac{\lambda_P(t)}{\lambda_m(t)} = \frac{E[Z \mid 0]}{E[Z \mid t]}$$

is also increasing and therefore relation (7) holds without additional assumptions.

4. COMPARISON WITH $\lambda_P(t)$

Theorem 1. *Let the failure rate* $\lambda(t,z)$ *in the mixing model (2) be differentiable with respect to both arguments and be ordered as*

$$\lambda(t,z_1) < \lambda(t,z_2), \quad z_1 < z_2, \forall z_1, z_2 \in [a,b], t \ge 0\,. \qquad (12)$$

Assume that conditional and unconditional expectations in relations (2) and (4), respectively, exist and finite for $\forall t \in [0,\infty)$. *Then:*

a) The mixture failure rate $\lambda_m(t)$ *bends down with time at least in a weak sense.*

b) If additionally, $\frac{\partial \lambda(t,z)}{\partial z}$ *is increasing in* t, *then* $\lambda_m(t)$ *bends down with time in a strong sense.*

Proof. It is clear that ordering (12) is equivalent to the condition that $\lambda(t,z)$ is increasing in z for each $t \ge 0$. In accordance with equations (2) and (4) and integrating by parts (Finkelstein, 2004):

$$\Delta\lambda(t) \equiv \int_a^b \lambda(t,z)[\pi(z) - \pi(z \mid t)]dz$$

$$= \lambda(t,z)[\Pi(z) - \Pi(z \mid t)]\,|_a^b - \int_a^b \lambda_z'(t,z)[\Pi(z) - \Pi(z \mid t)]dz$$

$$= \int_a^b -\lambda_z'(t,z)[\Pi(z) - \Pi(z \mid t)]dz > 0,\ \ t > 0\,, \tag{13}$$

where

$$\Pi(z) = P(Z \le z); \quad \Pi(z \mid t) = P(Z \le z \mid T > t)$$

and the term $\lambda(t,z)[\Pi(z) - \Pi(z \mid t)]\,|_a^b$ vanishes for $b = \infty$ as well. Inequality (13) and, therefore, the first part of the theorem follows from: $\lambda_z'(t,z) > 0$ and the following inequality:

$$\Pi(z) - \Pi(z \mid t) < 0, \forall t > 0, z \in (a,b)\,. \tag{14}$$

Inequality (14) can be interpreted as: **"the weakest populations are dying out first"**. This interpretation is widely used in specific cases, especially in the demographic literature (e.g., Vaupel, 2003). For obtaining (14), it is sufficient to prove that

$$\Pi(z \mid t) = \frac{\int_a^z \overline{F}(t,u)\pi(u)du}{\int_a^b \overline{F}(t,u)\pi(u)du}$$

is increasing in t, which will be also used for proving part b).

It is easy to see that $\Pi_t'(z \mid t) > 0$, if

$$\frac{\int_a^z \overline{F}_t'(t,u)\pi(u)du}{\int_a^z \overline{F}(t,u)\pi(u)du} > \frac{\int_a^b \overline{F}_t'(t,u)\pi(u)du}{\int_a^b \overline{F}(t,u)\pi(u)du} \tag{15}$$

As $\overline{F_t'}(t,z) = -\lambda(t,z)\overline{F}(t,z)$, it is sufficient to show that

$$B(t,z) \equiv \frac{\int_a^z \lambda(t,u)\overline{F}(t,u)\pi(u)du}{\int_a^z \overline{F}(t,u)\pi(u)du}$$

is increasing in z. Inequality $B_z'(t,z) > 0$ is equivalent to the following one:

$$\lambda(t,z)\int_a^z \overline{F}(t,u)\pi(u)du > \int_0^z \lambda(t,u)\overline{F}(t,u)\pi(u)du,$$

which is true, as $\lambda(t,z)$ is increasing in z.

Thus, due to additional assumption in b), the integrand in the end part of (13) is increasing and therefore $\Delta\lambda(t)$ as well, which immediately leads to the strong bending down property (6). ♦

Remark 1. Additional assumption b) means for the specific multiplicative model (8) that the baseline $\lambda(t)$ is an increasing function.

We will show now that a natural ordering for our mixing model is the likelihood ratio one. A somewhat similar reasoning can be found in Block *et al* (1993) and Shaked and Spizzichino (2001)). Let Z_1 and Z_2 be continuous nonnegative random variables with the same support and densities $\pi_1(z)$ and $\pi_2(z)$, respectively. Recall **(**Ross, 1996; Shaked and Shanthikumar, 1993) that Z_2 is smaller than Z_1 in the sense of likelihood ratio:

$$Z_1 \geq_{LR} Z_2 \ , \tag{16}$$

if $\pi_2(z)/\pi_1(z)$ is a decreasing function.

Definition 2. Let $Z(t), t \in [0,\infty)$ be a family of random variables indexed by parameter t (time) with probability density functions $p(z,t)$. We say that $Z(t)$ is decreasing in t in the sense of the likelihood ratio, if

$$L(z,t_1,t_2) = \frac{p(z,t_2)}{p(z,t_1)}$$

is decreasing in z for all $t_2 > t_1$.

The following simple result states that our family of conditional mixing random variables $Z \mid t, t \in [0,\infty]$ is decreasing in this sense:

Theorem 2. *Let the family of failure rates $\lambda(t,z)$ in the mixing model (2) be ordered as in relation (12).*
Then the family of random variables $Z \mid t \equiv Z \mid T > t$ is decreasing in $t \in [0,\infty)$ in the sense of the likelihood ratio.

Proof. In accordance with definition (3):

$$L(z,t_1,t_2) = \frac{\pi(z \mid t_2)}{\pi(z \mid t_1)} = \frac{\overline{F}(t_2,z)\int_a^b \overline{F}(t_1,z)\pi(z)dz}{\overline{F}(t_1,z)\int_a^b \overline{F}(t_2,z)\pi(z)dz}. \tag{17}$$

Therefore, monotonicity in z of $L(z,t_1,t_2)$ is defined by

$$\frac{\overline{F}(t_2,z)}{\overline{F}(t_1,z)} = \exp\left\{-\int_{t_1}^{t_2} \lambda(u,z)du\right\},$$

which, due to ordering (12), is decreasing in z for all $t_2 > t_1$.

5. DIFFERENT MIXING DISTRIBUTIONS

5.1. Likelihood ordering of mixing distributions

For the mixing model (2)-(3) consider two different mixing random variables Z_1 and Z_2 with probability density functions $\pi_1(z)$, $\pi_2(z)$ and cumulative distribution functions $\Pi_1(z), \Pi_2(z)$, respectively. Assuming some type of stochastic ordering for Z_1 and Z_2, we intend to arrive at a simple ordering of the corresponding mixture failure rates. It can be seen using simple examples that the 'usual' stochastic ordering (stochastic dominance) is too weak for this purpose. It was shown in the previous section that the likelihood ratio ordering is a natural one for the family of random variables $Z \mid t$ in our mixing model. Therefore, it seems reasonable to order Z_1 and Z_2 in this sense too.

Lemma. *Let*

$$\pi_2(z) = \frac{g(z)\pi_1(z)}{\int_a^b g(z)\pi_1(z)dz} \,, \tag{18}$$

where $g(z)$ is a decreasing function.

Then Z_1 is stochastically larger than Z_2:

$$Z_1 \geq_{st} Z_2 \qquad (\Pi_1(z) \leq \Pi_2(z),\, z \in [a,b]) \tag{19}$$

Proof.

$$\Pi_2(z) = \frac{\int_a^z g(u)\pi_1(u)du}{\int_a^b g(u)\pi_1(u)du} = \frac{\int_a^z g(u)\pi_1(u)du}{\int_a^z g(u)\pi_1(u)du + \int_z^b g(u)\pi_1(u)du}$$

$$= \frac{g^*(a,z)\int_a^z \pi_1(u)du}{g^*(a,z)\int_a^z \pi_1(u)du + g^*(z,b)\int_a^z \pi_1(u)du} \geq \int_a^z \pi_1(u)du = \Pi_1(z)\,, \tag{20}$$

where $g^*(a,z)$ and $g^*(z,b)$ are the mean values of the function $g(z)$ in the corresponding integrals. As this function decreases: $g^*(z,b) \leq g^*(a,z)$.

Remark 2. Equation (18) for decreasing $g(z)$ means that $Z_1 \geq_{LR} Z_2$ **,** and it is well known (see, e.g., Ross, 1996) that the likelihood ratio ordering implies the corresponding stochastic ordering. But we need the foregoing reasoning for deriving the following result:

Theorem 3. Let *relation (18), where* $g(z)$ *is a decreasing function hold, which means that* Z_1 *is larger than* Z_2 *in the sense of the likelihood ratio ordering.*

Assume that ordering (12) holds.

Then for $\forall t \in [0,\infty)$*:*

$$\lambda_{m1}(t) \equiv \frac{\int_a^b f(t,z)\pi_1(z)dz}{\int_a^b \overline{F}(t,z)\pi_1(z)dz} \geq \frac{\int_a^b f(t,z)\pi_2(z)dz}{\int_a^b \overline{F}(t,z)\pi_2(z)dz} \equiv \lambda_{m2}(t) \tag{21}$$

Proof. Inequality (21) means that the mixture failure rate, which is obtained for the stochastically larger (in the likelihood ratio ordering sense) mixing distribution, is larger for $\forall t \in [0,\infty)$ that the one obtained for the stochastically smaller mixing distribution.

We shall prove, firstly, that

$$\Pi_1(z \mid t) = \frac{\int_a^z \overline{F}(t,u)\pi_1(u)du}{\int_a^b \overline{F}(t,u)\pi_1(u)du} \leq \frac{\int_a^z \overline{F}(t,u)\pi_2(u)du}{\int_a^b \overline{F}(t,u)\pi_2(u)du} \equiv \Pi_2(z \mid t). \tag{22}$$

Indeed:

$$\frac{\int_a^z \overline{F}(t,u)\pi_2(u)du}{\int_a^b \overline{F}(t,u)\pi_2(u)du} = \frac{\int_a^z \overline{F}(t,u)\frac{g(u)\pi_1(u)}{\int_a^b g(u)\pi_1(u)du}du}{\int_a^b \overline{F}(t,u)\frac{g(u)\pi_1(u)}{\int_a^b g(u)\pi_1(u)du}du}$$

$$=\frac{\int_a^Z g(u)\overline{F}(t,u)\pi_1(u)du}{\int_a^b g(u)\overline{F}(t,u)\pi_1(u)du} \geq \frac{\int_a^Z \overline{F}(t,u)\pi_1(u)du}{\int_a^b \overline{F}(t,u)\pi_1(u)du},$$

where the last inequality follows using exactly the same argument, as in inequality (20) of the Lemma. Similar to (13) and taking into account relation (22):

$$\lambda_{m1}(t)-\lambda_{m2}(t)=\int_a^b \lambda(t,z)[\pi_1(z\,|\,t)-\pi_2(z\,|\,t)]dz$$

$$=\lambda(t,z)[\Pi_1(z\,|\,t)-\Pi_2(z\,|\,t)]\,|_a^b-\int_a^b \lambda'_z(t,z)[\Pi_1(z\,|\,t)-\Pi_2(z\,|\,t)]dz$$

$$=\int_a^b -\lambda'_z(t,z)[\Pi_1(z\,|\,t)-\Pi_2(z\,|\,t)]dz \geq 0,\ \ t>0, \tag{23}$$

where, similar to the proof of Theorem 1, the limit

$$\lim_{b\to\infty} \lambda(t,z)[\Pi_1(z\,|\,t)-\Pi_2(z\,|\,t)]\,|_a^b = 0\,.$$

was taken into account ♦

A starting point of Theorem 3 was equation (18) with a crucial assumption of a decreasing function $g(z)$. It should be noted, however, that this assumption can be rather formally justified directly by considering the difference $\Delta\lambda(t)=\lambda_{m1}(t)-\lambda_{m2}(t)$ and using definitions (2)-(3). The corresponding numerator (the denominator is positive) is transformed into a double integral:

$$\int_a^b \lambda(t,z)\overline{F}(t,z)\pi_1(z)dz\int_a^b \overline{F}(t,z)\pi_2(z)dz-\int_a^b \lambda(t,z)\overline{F}(t,z)\pi_2(z)dz\int_a^b \overline{F}(t,z)\pi_1(z)dz \tag{24}$$

$$=\int_a^b\int_a^b \overline{F}(t,u)\overline{F}(t,s)[\lambda(t,u)\pi_1(u)\pi_2(s)-\lambda(t,s)\pi_1(u)\pi_2(s)]duds$$

$$=\int_{\substack{a\\u>s}}^b\int_a^b \overline{F}(t,u)\overline{F}(t,s)[\pi_1(u)\pi_2(s)(\lambda(t,u)-\lambda(t,s))+\pi_1(s)\pi_2(u)(\lambda(t,s)-\lambda(t,u))]duds$$

$$= \int_{\substack{a \\ u>s}}^{b} \int_{a}^{b} \overline{F}(t,u)\overline{F}(t,s)(\lambda(t,u)-\lambda(t,s))(\pi_1(u)\pi_2(s)-\pi_1(s)\pi_2(u))duds\,.$$

Therefore, the final double integral is positive, if ordering (12) holds and $\pi_2(z)/\pi_1(z)$ is decreasing.

5.2. Ordering variances of mixing distributions

Let $\Pi_1(z)$ and $\Pi_2(z)$ be two mixing distributions with equal means. It follows from equation (9) that for the multiplicative model, which will be considered in this section: $\lambda_{m1}(0)=\lambda_{m2}(0)$. Intuitive considerations and reasoning based on the principle: “the weakest populations are dying out first” suggest that unlike (21), the mixture failure rates will be ordered as $\lambda_{m1}(t)<\lambda_{m2}(t)$ for all $t>0$ if, e.g., variance of Z_1 is larger than variance of Z_2. We will show that this is true for a specific case and that for a general multiplicative model the ordering holds only for sufficiently small time t. Therefore, a stronger condition on ordering ‘variabilities’ of Z_1 and Z_2 should be formulated.

For a meaningful specific example, consider the frailty model (8), where Z has a gamma distribution:

$$\pi(z)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}z^{\alpha-1}\exp\{-\beta z\};\, \alpha>0, \beta>0.$$

Substituting this density into relation (2):

$$\lambda_m(t)=\frac{\lambda(t)\int_0^\infty \exp\{-z\Lambda(t)\}z\pi(z)dz}{\int_0^\infty \exp\{-z\Lambda(t)\}\pi(z)dz},$$

where $\Lambda(t)=\int_0^t \lambda(u)du$ is a cumulative baseline failure rate. Computation of integrals results in:

$$\lambda_m(t)=\frac{\alpha\lambda(t)}{\beta+\Lambda(t)} \qquad (25)$$

Equations (25) can be written now in terms of $E[Z]$ and $Var(Z)$:

$$\lambda_m(t) = \lambda(t)\frac{E^2[Z]}{E[Z] + Var(Z)\Lambda(t)}, \tag{26}$$

which for the specific case $E[Z] = 1$ gives the widely used in demography result of Vaupel *et al* (1979):

$$\lambda_m(t) = \frac{\lambda(t)}{1 + Var(Z)\Lambda(t)}.$$

Using equation (26), we can compare mixture failure rates of two populations with different Z_1 an Z_2 on condition that $E[Z_2] = E[Z_1]$:

$$Var(Z_1) \geq Var(Z_2) \Rightarrow \lambda_{m1}(t) \leq \lambda_{m2}(t). \tag{27}$$

Intuitively it can be expected that this result could be valid for arbitrary mixing distributions in the multiplicative model. However, the mixture failure rate dynamics can be much more complicated even for this specific case and this topic needs further attention in the future research. A somewhat similar situation was observed in Finkelstein and Esaulova (2001): although the conditional variance $Var(Z \mid t)$ was decreasing in t for the multiplicative gamma-frailty model, a counter example was constructed for the case of the uniform mixing distribution in [0,1].

The following theorem shows that ordering of variances is a sufficient and necessary condition for ordering of mixture failure rates, but only for the initial time interval.

Theorem 4. *Let* Z_1 *and* Z_2 ($E[Z_2] = E[Z_1]$) *be two mixing distributions in the multiplicative model (8)- (9).*
Then ordering of variances

$$Var(Z_1) > Var(Z_2) \tag{28}$$

is a sufficient and necessary condition for ordering of mixture failure rates in the neighborhood of $t = 0$*:*

$$\lambda_{m1}(t) < \lambda_{m2}(t);\ t \in (0, \varepsilon), \tag{29}$$

where $\varepsilon > 0$ *is sufficiently small.*

Proof. Sufficient condition:

From results of Section 3:

$$\Delta\lambda(t) = \lambda_{m1}(t) - \lambda_{m2}(t) = \lambda(t)(E[Z_1 \mid t] - E[Z_2 \mid t), \tag{30}$$

$$E'_t[Z_i \mid t] = -\lambda(t)Var(Z_i \mid t) < 0, i = 1,2, t \geq 0, \tag{31}$$

where

$$E[Z_i \mid 0] \equiv E[Z_i], Var(Z_i \mid t) \equiv Var(Z_i). \tag{32}$$

As the means of mixing variables are equal, relation (30) for $t = 0$ reads: $\Delta\lambda(0) = 0$ and therefore the time interval in (29) is opened. Thus, if ordering (28) holds, ordering (29) follows immediately after considering the derivative of

$$\frac{\lambda_{m1}(t)}{\lambda_{m2}(t)} = \frac{E[Z_1 \mid t]}{E[Z_2 \mid t]}$$

at $t = 0$ and taking into account relations (31) and notation (32).

Necessary condition:

Similar to (24), the numerator of the difference $\Delta\lambda(t)$ is

$$\lambda(t)\int_a^b\int_a^b [\exp\{-\Lambda(t)(u+s)\}](u-s)\pi_1(u)\pi_2(s)duds,$$

where, as previously, $\Lambda(t) = \int_0^t \lambda(u)du$. After changing variables to

$x = (u+s)/2, y = (u-s)/2$, the double integral is transformed to the iterated integral and denoted by $G(t)$:

$$G(t)) \equiv \int_a^b \exp\{-2\Lambda(t)x\}\int_{-x}^x y\pi_1(x+y)\pi_2(x-y)dydx. \tag{33}$$

Denote the internal integral in (33) by $g(x)$. Then:

$$G(t) = \int_a^b [\exp\{-2\Lambda(t)x\}]g(x)dx.$$

On the other hand, coming back to initial variables of integration and taking into account that $\Lambda(0) = 0$:

$$G(0) = \int_a^b g(x)dx = \int_a^b\int_a^b (u-s)\pi_1(u)\pi_2(s)duds$$

$$= \int_a^b u\pi_1(u)du - \int_a^b u\pi_2(u)du = E[Z_1] - E[Z_2] = 0 .$$

Assume, firstly, that $\lambda(0) \neq 0$. As $G(0) = 0$, the function $G(t))$ is negative in the neighborhood of 0, if $G'0) < 0$:

$$G'(t) = -2\lambda(t)\int_a^b [\exp\{-2\Lambda(t)x\}]xg(x)dx ,$$

$$G'(0) < 0 \Rightarrow \int_a^b xg(x)dx > 0 .$$

If $\Delta\lambda(t) < 0, t \in (0,\varepsilon)$ (condition (29), then $G(t) < 0, t \in (0,\varepsilon)$, and taking into account that

$$\int_a^b xg(x)dx = \int_a^b\int_a^b \frac{u+s}{2}(u-s)\pi_1(s)\pi_2(s)duds$$

$$= \frac{1}{2}\int_a^b\int_a^b (u^2 - s^2)\pi_1(u)\pi_2(s)duds = \frac{1}{2}(Var(Z_1) - Var(Z_2)),$$

we arrive at ordering (28).

Similar considerations are valid for $\lambda(0) = 0$. The function $G(t))$ is negative in this case in the neighborhood of 0, if $G''0) < 0$. As

$$G''(0) = -2\lambda'(0)\int_a^b xg(x)dx$$

and $\lambda'(0) > 0$ (as $\lambda(t) > 0, t > 0$ and $\lambda(0) = 0$), the foregoing reasoning which was used for the case $\lambda(0) \neq 0$, also takes place.♦

A trivial but important consequence of this theorem is:

Corollary. *Let mixtures failure rate ordering (29) hold for* $t \in (0,\infty)$. *Then inequality (28) holds.*

Remark 3. It follows from Theorem 4, that ordering of variances of mixing distributions is a too weak condition for obtaining ordering of mixture failure rates for all $t > 0$. As it was mentioned, an effect of 'variability' of a mixing distribution on the shape of the mixture failure rate can be quite complex. We have explored several possibilities of stronger assumptions and came to the *conjecture* (to be proved yet) for the case of an infinite support ($a = 0, b = \infty$) that the following sufficient condition (along with ordering (12) and condition b) of Theorem 1 and $E[Z_2] = E[Z_1]$) will result in ordering (29) for all $t > 0$:

Let mixing distributions $\Pi_1(z)$ *and* $\Pi_2(x)$ *have only one crossing point* $c : \Pi_1(z) \geq \Pi_2(z), z < c$ *and* $\Pi_1(z) \leq \Pi_2(z), z \geq c$.

It can be shown that this condition implies the convex order: $Z_1 \geq_{cx} Z_2$ (Kaas *et al,*1994), which 'gives more variability' to Z_1 than to Z_2.

It follows from equations similar to (23) that:

$$\int_0^\infty [\overline{\Pi}_2(z \mid t) - \overline{\Pi}_1(z \mid t)]dz \geq 0 \Rightarrow \lambda_{m1}(t) \leq \lambda_{m2}(t), t \geq 0 .$$

The left hand side of this relation can be hopefully proved using the one-crossing property of mixing distributions.

6. CONCLUDING REMARKS

The mixture failure rate is bent down due to "the weakest populations are dying out first" effect, mathematically described in Section 4. This should be taken into account when analyzing the failure data for heterogeneous populations.

A family of conditional mixing random variables $(Z \mid t)$ is decreasing in $t \in [0, \infty)$ in the sense of the likelihood ratio. This is a natural ordering for mixing random variables in the problem under consideration. Therefore, when different mixing random variables are ordered in the sense of the likelihood ratio, the mixture failure rates are ordered accordingly.

Mixing distributions with equal expectations and different variances can lead to the corresponding ordering for mixture failure rates in $[t, \infty)$ in some specific cases. For the general mixing distribution in the multiplicative model, however, this ordering is guaranteed only for sufficiently small t. On the other hand, the convex order (in fact, a stronger condition) in mixing distributions can still hopefully result in the desired ordering in mixture failure rates, and this is our conjecture.

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