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Abstract

Burn-in is a widely used engineering method which is adopted to eliminate defective items before they are shipped to customers or put into the field operation. In the studies of burn-in, the assumption of bathtub shaped failure rate function is usually employed and optimal burn-in procedures are investigated. In this paper, however, we assume that the population is composed of two ordered subpopulations and optimal burn-in procedures are studied in this context. Two types of risks are defined and an optimal burn-in procedure, which minimizes the weighted risks is studied. The joint optimal solutions for the optimal burn-in procedure, which minimizes the mean number of repairs during the field operation, are also investigated.

Keywords : Mixed population, ordered subpopulations, main population, stochastic order, weighted risk, burn-in procedure, minimal repair

ACRONYMS

Cdf cumulative distribution function
FR instantaneous failure rate (function)
pdf probability density function
r.v. random variable
Sf survivor function

NOTATION

X_S lifetime of the strong component, $X_S \geq 0$; a r.v.

X_W lifetime of the weak component, $X_W \geq 0$; a r.v.

$F_S(t), r_S(t), \Lambda_S(t)$ Cdf, FR and cumulative FR of X_S

$F_W(t), r_W(t), \Lambda_W(t)$ Cdf, FR and cumulative FR of X_W

$\rho(t)$ the scale transformation function

$p, 1-p$ the proportions of strong and weak subpopulations in the population, respectively

b burn-in time

n the critical number of failures during burn-in

F_1 the event that the item passes the burn-in process

F_2 the event that the item is eliminated by the burn-in process

S the event that the item is from the strong subpopulation

W the event that the item is from the weak subpopulation

w_1, w_2 weights of two types of risks

$N(b)$ the number of minimal repairs during the burn-in time $[0, b)$; a r.v.

τ given mission time

t^*, t^{**} the first and second wear out points, respectively, when the FR is eventually increasing

s^* the uniform upper bound for optimal burn-in time

1. Introduction

Burn-in is a method of 'elimination' of initial failures in field usage. To burn-in a component or a system means to subject it to a period of simulated use prior to actual operation. Due to the high FR at the early stages of component's life, burn-in has been widely accepted as an effective method of screening out failures before systems are actually used in field operations. An introduction to this important area of reliability engineering can be found in Jensen and Petersen (1982) and Kuo and Kuo (1983).

If burn-in procedure is applied for 'too long', then the items with 'high reliability' can also be eliminated by burn-in or their remaining lifetimes can be substantially decreased. On the other hand, if burn-in procedure is too short in time, then the items with 'low reliability' can still remain in the population, which results in frequent failures at the early stages of component's usage. As burn-in is usually costly, one of the major problems is to define the duration of this procedure. The best time to stop the burn-in process for a given criterion is called the optimal burn-in time. In the literature, various cost structures have been proposed, and the corresponding problem of finding the optimal burn-in time has been considered. See, for example, Nguyen and Murthy (1982), Clarotti and Spizzichino (1991), Mi (1994a, 1996, 1997) and Cha (2000). Some other performance-based criteria, for example, the mean residual life, the reliability for the given mission time, or the mean number of failures, have been also considered to determine the optimal burn-in time (See also Mi (1991, 1994b), Block et al. (1994, 2002)). An excellent survey of research in this area can be found in Block and Savits (1997).

In most papers mentioned above, the burn-in procedures have been studied under the assumption that the FR of the system follows the traditional bathtub shape. However, recently there has been much research on the shape of the FR for mixtures of distributions. The IFR, bathtub shape, the modified bathtub shape (first, increasing and then bathtub) and some other shapes can occur as specific cases of mixing (see, e.g., Jiang and Murphy (1995), Gupta and Warren (2001), Block et al. (2003a, 2003b) and Klutke et al. (2003)). It was stated also that the bathtub shaped FR describes only up to 15% of applications (See, e.g., Kececioglu and Sun (1995)). Thus, the assumption of the bathtub-shaped FR can be sometimes considered as rather superficial.

In this paper, a new burn-in approach for repairable items is proposed and optimal burn-in procedure is investigated without assuming initially the bathtub shape of a population FR. We consider the mixed population composed of two ordered subpopulations – the subpopulation of strong items (items with 'normal' lifetimes) and that of weak items (items with shorter lifetimes). Based on the information obtained during the burn-in procedure, items are classified into two groups: one class of items, which is considered to belong to the strong subpopulation and the other class of items that is believed to belong to the weak subpopulation. Then the items belonging to the second class are eliminated (discarded) and only the remaining items are considered to be suitable for the field operation.

As, e.g., in Mi (1991, 1994b), Block et al. (1994, 2002), we study an optimal burn-in which optimizes the defined performance-based criterion. In the first part of the paper, we consider two types of risks – (i) the risk that a strong component will be eliminated during burn-in and (ii) the risk that a weak component will pass the burn-in procedure. Optimal burn-in, which minimizes the weighted average of these risks, is investigated. The second part deals with optimal burn-in which minimizes the mean number of failures during the given mission time. It should be emphasized that the obtained optimal burn-in procedure (which minimizes the mean number of repairs during field usage) is suggested mainly for the case when the field mission is very important and the failures (even minimally repaired) during this mission are very undesirable (e.g., military missions). The costs incurred during burn-in are usually not so important in this case.

2. Stochastically Ordered Subpopulations and Mixed Distributions

Due to the high initial FR that often occurs in the early stages of component's life, burn-in has been considered as an essential procedure for revealing early failures. In Jensen and Petersen (1982), based on various sets of field data, it is observed that the population of produced items is composed of two subpopulations - the strong subpopulation with normal lifetimes and the weak subpopulation with shorter lifetimes. In practice, weak items can be produced along with strong items due to, for example, defective resources and components, human errors, unstable production environment caused by uncontrolled significant quality factors, and, etc. Mixture of these two subpopulations often results in a bimodal distribution as illustrated in Jensen and Petersen (1982). According to these authors, e.g., the infant mortality period of the life cycle that exhibits high FR, results from failures in a weak subpopulation of a bimodal lifetime distribution. This can also be well understood if we observe the fact that weak items tend to fail earlier than strong items. In other words: the weakest populations are dying out first (Finkelstein, 2008). Thus, in view of this context, it can be stated that one of the main purposes of the burn-in procedure is to eliminate the weak subpopulation from the mixed population.

Having in mind these considerations, we assume in our paper that the population is a mixture of two ordered subpopulations – the strong subpopulation and the weak subpopulation.

Let the lifetime of a component from the strong subpopulation be denoted by X_S and its absolutely continuous Cdf be $F_S(t)$. Similarly, the lifetime and the Cdf of a weak component is denoted by X_W and $F_W(t)$, respectively. It is reasonable to assume that these lifetimes are ordered as:

$$X_W \le_{st} X_S , \tag{1}$$

which means that (see, e.g., Ross (1996))

$$F_S(t) \le F_W(t), t \ge 0 . \tag{2}$$

These inequalities define a general stochastic ordering between two r.v.'s. Note that, since a Cdf of an absolutely continuous r.v. is a continuous function that increases from 0 to 1, the relationship defined in (2) is equivalent to the following equation:

$$F_W(t) = F_S(\rho(t)), \forall t \geq 0, \tag{3}$$

where $\rho(t)$ is non-decreasing, $\rho(t) \geq t, \forall t \geq 0$, and $\rho(0) = 0$. Throughout this paper, we assume the stochastic ordering (2)-(3). Let $r_S(t)$ be the FR which corresponds to X_S. Then, the FR $r_W(t)$ for X_W, as follows from (3), is given by

$$r_W(t) = \rho'(t) r_S(\rho(t)). \tag{4}$$

Another important ordering in reliability applications is the FR ordering, which is defined as

$$r_S(t) \leq r_W(t), t \geq 0. \tag{5}$$

It can be easily seen that the ordering (5) implies (1), and therefore equation (3) also holds. A practical specific case of (5) is the proportional hazards model that can be defined in our case as

$$r_W(t) = \rho\, r_S(t), t \geq 0, \tag{6}$$

where $\rho > 1$. From a practical point of view, relationship (6) constitutes a reasonable model for defining the subpopulations of interest. For practical applications, when exponential distribution is assumed, (6) turns to:

$$r_W = \rho\, r_S.$$

We assume that the proportion of items from the strong subpopulation in the total population is p. Then the Cdf of the total population is given by the following mixture:

$$G(t) = pF_S(t) + (1-p)F_S(\rho(t)),$$

whereas the proportional hazards model (6) results in

$$G(t) = pF_S(t) + (1-p)(1-(\overline{F}_S(t))^{\rho}),$$

where $\overline{F} \equiv 1 - F$.

Furthermore, assume that items are repairable and undergo minimal repair upon failure (See also Cha (2000, 2006)).

3. Optimal Burn-in Procedure for Minimizing Weighted Risks

In this paper, we adopt the following Burn-in Procedure.

•Burn-in Procedure:

*The item is burned-in during $(0, b]$ and if the number of **minimally repaired** failures during burn-in process $N(b)$ satisfies $N(b) \leq n$ then the item is considered as one from the strong subpopulation and put into field operation; otherwise the item is considered as one from the weak subpopulation and is discarded.*

Before starting with quantification of the described burn-in procedure, it is reasonable to clarify the term “minimal repair” for our settings of this and the following sections.

•Minimal repair

Minimal repair is usually defined in the classical sense as the repair that brings an item to the *statistically* identical state it had just prior to the failure (Barlow and

Proschan, 1975). For the item with the Cdf $F(t)$ that had failed and was instantaneously minimally repaired at time a, it means that the time to the next failure is distributed as $(F(t+a)-F(a))/(1-F(a))$, which is equal to the Cdf of the corresponding remaining lifetime for a nonrepairable item. This type of minimal repair is sometimes called a *statistical minimal repair*. (Arjas and Norros, 1989; Finkelstein, 1992) to emphasize the repair to the mentioned above statistically identically state, but usually the term "statistical" is omitted.

Minimal repair in heterogeneous populations is not so unambiguous as in the homogeneous case (Finkelstein, 2004, 2008). In the case under consideration, we have a mixed (heterogeneous) infinite population described by the mixture distribution $G(t)$. This means that formally, in accordance with the classical definition, the time to the next failure should be distributed as $(G(t+a)-G(a))/(1-G(a))$. The only theoretical possibility to perform this operation is to replace the failed item by another item from our population that had functioned for the same time but did not fail. It is obvious that it is practically impossible to achieve this ideal statistical minimal repair in reality. On the other hand, if it would be possible, then e.g., for the case of proportional hazards with constant FRs, when the mixture FR is decreasing in $[0,\infty)$ (Barlow and Proschan, 1975), the probability that after each repair we are choosing an item from the strong population is increasing to 1 as $t \to \infty$. This means that when the mixture FR is decreasing, an "ideal burn-in procedure" without discarding can be performed. When, e.g., the mixture FR has a bathtub shape, as it was already mentioned, different optimal burn-in procedures can be performed, but again, one cannot execute the corresponding statistical minimal repair in practice.

Our case is different. At $t=0$ an item from a mixed population is chosen and put into operation. Upon failure at $t=a$ it is minimally repaired, etc. An item that does not meet our burn-in criterion is discarded. Therefore the main goal is to classify the mixed populations into the weak and strong populations. We assume that the corresponding minimal repair is, in fact, a *physical minimal repair* (Finkelstein, 1992) in the sense that a 'physical operation' of repair (not a replacement as above) brings an item in the state which is 'statistically identical' to the state it had just prior the failure. Note that, obviously, we do not know whether an item is 'strong' or 'weak'. On the other hand, the described operation in some sense 'keeps a memory of that': if it is, e.g., 'strong', the time to the next failure is distributed as $(F_S(t+a)-F_S(a))/(1-F_S(a))$, etc. An example of this 'physical operation' is when a small realized defect (fault) is corrected upon failure, whereas the number of the possible inherent defects in the item is large. In practice, physical minimal repair of the described type can be usually performed and therefore our assumption is quite realistic.

By various practical reasons, total burn-in time is generally limited. Therefore, in this section, we assume that the burn-in time is fixed as b. Then the above burn-in procedure can be defined in terms of n and we find an optimal burn-in procedure n^* which minimizes the appropriately defined risk.

For description of related risks, define the following four events:

-Event F_1: the item passes the burn-in process;

-Event F_2: the item is eliminated by the burn-in process;

-Event S : the item is from the strong subpopulation;

-Event W : the item is from the weak subpopulation.

Then

$$P(F_2 \mid S) = 1 - P(F_1 \mid S) \text{ and } P(F_1 \mid W) = 1 - P(F_2 \mid W).$$

Note that $P(F_2 \mid S)$ is, the so-called, the risk of the first order (the probability that the strong component is eliminated) and $P(F_1 \mid W)$ is the risk of the second order (the probability that the weak component had passed the burn-in). Therefore our goal is to minimize these risks. Basically we have 3 options:

Firstly, we minimize the first risk $P(F_2 \mid S)$ not taking into account the second risk. Then this problem is equivalent to maximizing $P(F_1 \mid S)$. In accordance with the well-known property, the process of minimal repairs is the corresponding nonhomogeneous Poisson process (NHPP). Therefore, taking into consideration our reasoning with respect to minimal repair:

$$P(F_1 \mid S) = \sum_{i=0}^{n} \frac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!},$$

where $\Lambda_S(t) \equiv \int_0^t r_S(u)du$ is the corresponding cumulative FR. Obviously, the maximum is achieved when $n = \infty$. This is an intuitively clear trivial solution, as we are not concerned about the other risk and 'are free' to minimize $P(F_2 \mid S)$. Therefore, this value can be as close to 0 as we wish. In practice, sometimes this setting can occur but, in that case, the optimal n^* should be defined via the corresponding restrictions on the allocated burn-in resources, burn-in costs, etc.

Secondly, we minimize $P(F_1 \mid W)$ not taking into account the first risk. Then this problem is equivalent to maximizing $P(F_2 \mid W)$. In this case,

$$P(F_2 \mid W) = 1 - \sum_{i=0}^{n} \frac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!},$$

where, as follows from (4):

$$\Lambda_W(t) \equiv \int_0^t r_W(u)du = \int_0^{\rho(t)} r_S(u)du = \Lambda_S(\rho(t)). \tag{7}$$

The maximum is achieved when $n = 0$. The corresponding value is

$$P_{n=0}(F_2 \mid W) = 1 - e^{-\Lambda_S(\rho(b))},$$

which means that the second order risk in this case is equal to the probability that an item from the weaker population will survive the burn-in process without any failures, which makes a perfect sense.

The previous two options were illustrative, as their settings are usually non-realistic. The appropriate approach should take into account both types of risk. Therefore, it is reasonable to consider minimization of the weighted risks:

$$\Psi(n) \equiv w_1 P(F_2 \mid S) + w_2 P(F_1 \mid W)$$
$$= 1 - [w_1 P(F_1 \mid S) + w_2 P(F_2 \mid W)],$$

where w_1 and w_2 are the weights satisfying $w_1 + w_2 = 1$. When $w_1 = 1$, $w_2 = 0$, we arrive at the first considered option, whereas the case $w_1 = 0$, $w_2 = 1$ corresponds to the second one. Furthermore, if $w_1 = w_2 = 1/2$, then we should minimize the sum of two risks $[P(F_2 \mid S) + P(F_1 \mid W)]$ or, equivalently, maximize the sum of the probabilities of correct decisions $[P(F_1 \mid S) + P(F_2 \mid W)]$.

Let n^* be the optimal burn-in procedure that satisfies

$$\Psi(n^*) = \min_{n \ge 0} \Psi(n). \tag{8}$$

This value is given by the following theorem:

Theorem 1. *Let* $0 < w_i < 1$, $i = 1,2$, *and* n^* *be the nonnegative integer which satisfies* (8). *If*

$$\frac{(\Lambda_S(\rho(b)) - \Lambda_S(b)) + (\ln w_1 - \ln w_2)}{\ln\left(\dfrac{\Lambda_S(\rho(b))}{\Lambda_S(b)}\right)} < 1,$$

then the optimal n^* *is given by* $n^* = 0$, *otherwise* n^* *is the largest integer which is less than or equal to*

$$\frac{(\Lambda_S(\rho(b)) - \Lambda_S(b)) + (\ln w_1 - \ln w_2)}{\ln\left(\dfrac{\Lambda_S(\rho(b))}{\Lambda_S(b)}\right)}.$$

Corollary 1. *When the specific proportional hazard model (6) holds, the cumulative FRs in (7) can be expressed in a more explicit way:*

$$\Lambda_W(t) = \int_0^t r_W(u)du = \rho\int_0^t r_S(u)du = \rho\Lambda_S(t).$$

In this case, if

$$\frac{(\rho - 1)\Lambda_S(b) + (\ln w_1 - \ln w_2)}{\ln \rho} < 1,$$

then the optimal n^* *is given by* $n^* = 0$, *otherwise* n^* *is the largest integer which is less than or equal to*

$$\frac{(\rho - 1)\Lambda_S(b) + (\ln w_1 - \ln w_2)}{\ln \rho}.$$

Example 1. Suppose that the FR of the strong subpopulation is given by

$$r_S(t) = \begin{cases} 1, 0 \le t \le 10 \\ t - 9, t > 10 \end{cases},$$

and $\rho(t)$ in (3) is given by $\rho(t) = 5t, t \ge 0$. The corresponding FR of the weak subpopulation is

$$r_W(t) = \begin{cases} 5, 0 \le t \le 2 \\ 25t - 45, t > 2 \end{cases}$$

and therefore the FR order between X_S and X_W holds. Suppose further that the burn-in time for this mixed population is given by $b = 1.0$ and $w_1 = 0.8$, $w_2 = 0.2$. Then

$$\frac{(\Lambda_S(\rho(b)) - \Lambda_S(b)) + (\ln w_1 - \ln w_2)}{\ln\left(\frac{\Lambda_S(\rho(b))}{\Lambda_S(b)}\right)} = 3.34 .$$

Therefore, the optimal burn-in procedure is determined by $n^* = 3$.

4. Optimal Burn-in Procedure for Minimizing Expected Number of Minimal Repairs

In this section, we discuss optimal burn-in that minimizes the mean number of minimal repairs during the mission time τ. We consider the same burn-in procedure as in Section 3, but now it is characterized by both b and n (i.e., b and n are burn-in parameters).

Observe that

$$P(F_1) = P(F_1 \mid S) \times P(S) + P(F_1 \mid W) \times P(W)$$
$$= \left(\sum_{i=0}^{n} \frac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}\right) \times p + \left(\sum_{i=0}^{n} \frac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}\right) \times (1-p) .$$

$$P(S \mid F_1) = \frac{P(S \cap F_1)}{P(F_1)} = P(F_1 \mid S) \times P(S) / P(F_1)$$

$$= \frac{\left(\sum_{i=0}^{n} \frac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}\right) \times p}{\left(\sum_{i=0}^{n} \frac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}\right) \times p + \left(\sum_{i=0}^{n} \frac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}\right) \times (1-p)} .$$

$$P(W \mid F_1) = \frac{\left(\sum_{i=0}^{n} \frac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}\right) \times (1-p)}{\left(\sum_{i=0}^{n} \frac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}\right) \times p + \left(\sum_{i=0}^{n} \frac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}\right) \times (1-p)} .$$

Let $\Psi(b,n)$ be the mean number of minimal repairs during the mission time τ in field operation given that the duration of burn-in is equal to b and that the rejection number is n. Then, in accordance with the above formulas and noting once again that the mean number of minimal repairs is equal to the cumulative intensity function of the corresponding NHPP,

$$\Psi(b,n) = \left(\Lambda_S(b+\tau) - \Lambda_S(b)\right)$$

$$\times \frac{\left(\sum_{i=0}^{n} \frac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}\right) \times p}{\left(\sum_{i=0}^{n} \frac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}\right) \times p + \left(\sum_{i=0}^{n} \frac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}\right) \times (1-p)}$$

$$+ \left(\Lambda_S(\rho(b+\tau)) - \Lambda_S(\rho(b))\right)$$

$$\times \frac{\left(\sum_{i=0}^{n} \frac{\left(\Lambda_S(\rho(b))\right)^i e^{-\Lambda_S(\rho(b))}}{i!}\right) \times (1-p)}{\left(\sum_{i=0}^{n} \frac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}\right) \times p + \left(\sum_{i=0}^{n} \frac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}\right) \times (1-p)} . \quad (11)$$

The objective is to find optimal (b^*, n^*) which satisfies

$$\Psi(b^*, n^*) = \min_{b \geq 0, n \geq 0} \Psi(b, n) . \quad (12)$$

In order to find the joint optimal solution defined in (12), we follow the procedure similar to that given in Mi (1994) and Cha (2000), where the two-dimensional optimization problems of finding the optimal burn-in time b^* and the age-replacement policy T^* that minimize the long-run average cost rate $c(b,T)$ are considered.

At the first stage, we fix the burn-in time b and find optimal $n^*(b)$ that satisfies

$$\Psi(b, n^*(b)) = \min_{n \geq 0} \Psi(b, n) . \quad (13)$$

At the second stage, we search for b^* that satisfies

$$\Psi(b^*, n^*(b^*)) = \min_{b \geq 0} \Psi(b, n^*(b)) .$$

Then the joint optimal solution is given by $(b^*, n^*(b^*))$, since the above procedure implies that

$$\begin{aligned} \Psi(b^*, n^*(b^*)) &\leq \Psi(b, n^*(b)), \text{ for all } b \geq 0, \\ &\leq \Psi(b, n), \text{ for all } b \geq 0,\ n \geq 0 . \end{aligned}$$

As in Mi (1994) and Cha (2000), in this case, if an uniform upper bound (with respect to n) could be found, then the optimization procedure would be much simpler.

Following the procedure described above, first find optimal $n^*(b)$ satisfying (13) for each fixed b. For this purpose, we need to state the following lemma which will be used for obtaining the optimal $n^*(b)$:

Lemma 1 (Mi, 2002). Suppose that $a_i \geq 0, i \geq 1,$ and $b_i > 0, i \geq 1.$ Then

$$\min_{1 \leq i \leq n} \frac{a_i}{b_i} \leq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \leq \max_{1 \leq i \leq n} \frac{a_i}{b_i} ,$$

where the equality holds if and only if all the $a_i / b_i, i \geq 1$, are equal.

The optimal value $n^*(b)$ is defined by the following theorem.

Theorem 2. *For a given fixed* $b \geq 0$, *let the following inequality:*

$$\left(\Lambda_S(b+\tau)-\Lambda_S(b)\right) \leq \left(\Lambda_S(\rho(b+\tau))-\Lambda_S(\rho(b)\right)$$

hold. Then the optimal $n^*(b)$ *is given by* $n^*(b)=0$, *whereas* $n^*(b)=\infty$ *corresponds to the opposite sign of the inequality.*

Remark 1. When the FR ordering (5) holds, the first inequality in Theorem 2 corresponds to

$$\left(\Lambda_S(b+\tau)-\Lambda_S(b)\right) \leq \left(\Lambda_W(b+\tau)-\Lambda_W(b)\right),$$

which is always obviously satisfied. For the specific case (6), it leads to

$$\int_b^{b+\tau} r_S(u)du = \left(\Lambda_S(b+\tau)-\Lambda_S(b)\right) \leq \left(\Lambda_W(b+\tau)-\Lambda_W(b)\right) = \rho \int_b^{b+\tau} r_S(u)du .$$

Remark 2. The result $n^*(b)=\infty$ (Theorem 2, Case 2) implies that after the burn-in time b with minimal repair every item is put into the field operation regardless of the number of failures during burn-in. This burn-in procedure is the same as that proposed in Cha (2000). Case 2 can obviously occur when the cumulative FR in $[0,b)$ for the strong subpopulation is smaller than that for the weak subpopulation, whereas the reverse ordering holds for the interval $[b,b+\tau)$ (e.g., when $r_S(t)$ has a decreasing part). In this case, the 'quality' of items *after burn-in* in the weak subpopulation is better than that in the strong subpopulation. Therefore, the burn-in procedure should leave all weak items in the population, which results in $n^*(b)=\infty$.

Consider now obtaining an uniform upper bound (with respect to n), i.e., we will find an upper bound for b^* denoted by s^*, such that,

$$\min_{0\leq b\leq s^*} \Psi(b,n) < \min_{b>s^*} \Psi(b,n),$$

for all fixed $n \geq 0$.

The following result gives an uniform upper bound for the optimal burn-in time b^*, but first we need to define the notion of the eventually (ultimately) increasing function (Gurland, Sethuraman, 1995, Mi, 2003).

Definition 1. The FR $r(x)$ is eventually increasing if there exists $0 \leq x_0 < \infty$ such that $r(x)$ strictly increases in $x > x_0$.

For the eventually increasing FR $r(x)$, the first and the second wear-out points t^* and t^{**} are defined in Mi (2003) as

$$t^* = \inf\{t \geq 0 : r(x) \text{ is non-decreasing in } x \geq t\} ,$$
$$t^{**} = \inf\{t \geq 0 : r(x) \text{ strictly increases in } x \geq t\} .$$

Observe that the eventually increasing FR can be constant in parts of the interval (t^*,t^{**}), whereas $t^* = t^{**}$ is obviously a specific case.

Theorem 3. *Suppose that*
(i) $r_S(t)$ *is eventually increasing with the first wear-out point* t^*, *the second wear-out point* t^{**} *and* $\lim_{t\to\infty} r_S(t) = \infty$;
(ii) $\rho(t)$ *is weak (i.e., not necessarily strictly) convex function.*
Then $s^* \in [t^*, \infty)$, *defined as*

$$s^* = \inf\{b' > t^* \mid \int_{\rho(t^*)}^{\rho(t^*+\tau)} r_S(u)du < \int_b^{b+\tau} r_S(u)du, \forall b > b'\}, \tag{14}$$

is the uniform upper bound for the optimal burn-in time b^*.

Example 2. Let the FR of the strong subpopulation be given by

$$r_S(t) = \begin{cases} 1, 0 \le t \le 6, \\ t-5, t \ge 6. \end{cases} \tag{15}$$

Then $t^* = 0$ and $t^{**} = 6$. Assume that $\rho(t) = 2t, t \ge 0$, $\tau = 2.0$. Then $\int_{\rho(t^*)}^{\rho(t^*+\tau)} r_S(u)du = 4.0$ and it is easy to see that $s^* = 6.0$.

It follows from Theorem 2 that, for each b, either $n^*(b) = 0$ or $n^*(b) = \infty$. Moreover, with the uniform upper bound s^* defined by Theorem 3, we can search for b^* which minimizes $\Psi(b, n^*(b))$ in the reduced interval $[0, s^*]$. Then Theorems 2-3 imply that the joint optimal solution is given by $(b^*, n^*(b^*))$. Based on these facts, the optimization procedure can be summarized as follows:

< Optimization Procedure (Algorithm)>

(Stage1)
Fix $0 \le b \le s^*$. If $\Lambda_S(b+\tau) - \Lambda_S(b) \le \Lambda_S(\rho(b+\tau)) - \Lambda_S(\rho(b))$ then $n^*(b) = 0$; otherwise $n^*(b) = \infty$.

(Stage2)
Find b^* which satisfies

$$\Psi(b^*, n^*(b^*)) = \min_{0 \le b \le s^*} \Psi(b, n^*(b)).$$

(Joint Optimal Solution)
Then the two dimensional optimal solution is given by $(b^*, n^*(b^*))$.

Example 3. Consider the setting of Example 2 and suppose that the FR $r_S(t)$ is given by (15). Furthermore, as in Example 2, assume that $\rho(t) = 2t, t \ge 0$, $\tau = 2.0$ and the proportion of strong subpopulation is $p = 0.9$. Then, as given in Example 2, the uniform upper bound s^* is given by $s^* = 6.0$. Thus, in order to find the joint optimal solution (b^*, n^*), we follow the optimization procedure described above. However, in this case, since $\rho(t)$ is a convex function and $r_S(t)$ is a non-decreasing function, the inequality

$$\Lambda_S(b+\tau)-\Lambda_S(b)\le\Lambda_S(\rho(b+\tau))-\Lambda_S(\rho(b)),\forall b\ge 0. \tag{16}$$

always holds. Thus $n^*(b)=0$, for all $b\ge 0$. Then the optimal solution $(b^*,n^*(b^*))$ is given by $(b^*,0)$, where b^* is the value which satisfies

$$\Psi(b^*,0)=\min_{0\le b\le 6.0}\Psi(b,0).$$

The graph for $\Psi(b,0)$ is given in Figure 1 along with the graph for $\Psi(b,\infty)$.

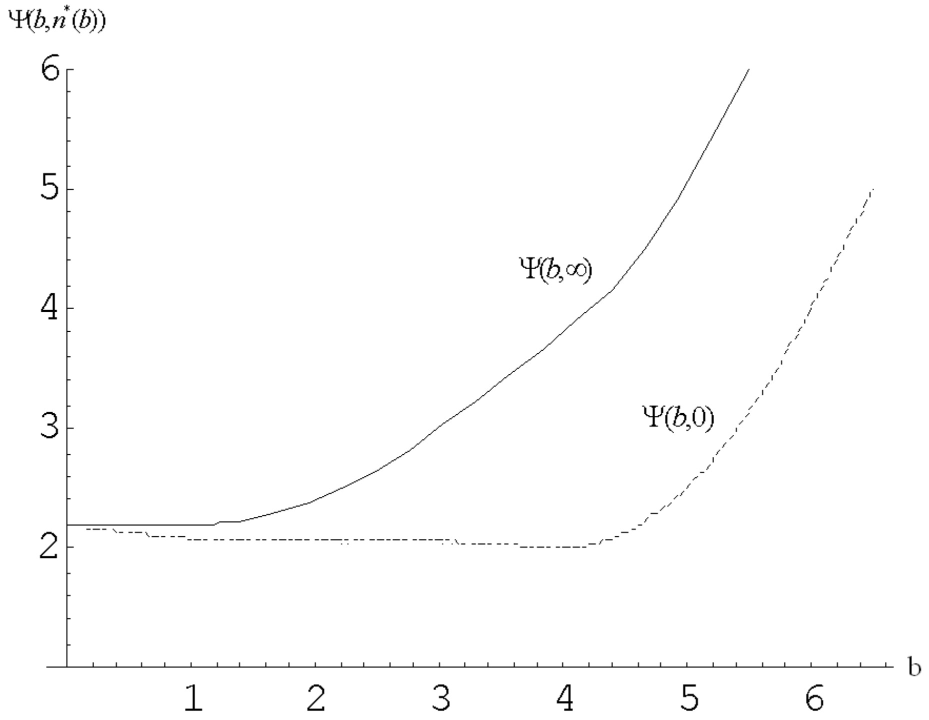

Figure 1. Function of Mean Number of Minimal Repairs

By a numerical search, it has been obtained that $b^*=4.048$ and minimum value of $\Psi(b,n)$ at optimal point $(b^*,n^*)=(4.048,\,0)$ is given by $\Psi(4.048,0)=2.00396$. Note that, by Theorem 1, the minimum value of $\Psi(b,n)$ for each fixed n is given by $\Psi(b,0)$ or $\Psi(b,\infty)$. In this specific example, due to inequality (16), $\Psi(b,0)\le\Psi(b,\infty)$.

The discussion based on the specific setting of Example 3 ($\rho(t)$ is a convex function and $r_S(t)$ is a non-decreasing function) can be summarized by the following corollary:

Corollary 2. *Suppose that*
(i) $r_S(t)$ *is eventually increasing with the first wear-out point* $t^*=0$*, the second wear-out point* t^{**} *and* $\lim_{t\to\infty} r_S(t)=\infty$*;*
(ii) $\rho(t)$ *is a weak convex function.*

Then the joint optimal solution is given by $(b^*,0)$, *where* b^* *is the value which satisfies*

$$\Psi(b^*,0) = \min_{0\le b\le s^*} \Psi(b,0),$$

and s^* *is the uniform upper bound given in (14).*

5. **Concluding Remarks**

In most papers dealing with optimal burn-in procedures the assumption on the shape of the FR of population has been made, e.g., the bathtub-shaped FR, the eventually increasing FR, etc. In our paper, on the contrary, we consider the mixture of two ordered subpopulations which can have different shapes of the population FR, and this can be considered as a more realistic and practical setting.

Two types of risks are considered and the optimal burn-in procedure defined by the optimal critical value for the number of failures during burn-in is studied. We also consider another type of the burn-in procedure, which is characterized by both burn-in time and the critical number of failures during burn-in. The optimal solution that minimizes the mean number of minimal repairs during field operation has been investigated in this case. Some numerical examples which illustrate the utility of the obtained results are also given.

Since the assumptions on the parametric model proposed in this paper (e.g., on functions $\rho(t)$ and $r_S(t)$) are quite general and not too restrictive, the obtained results can be used in many real applications. Furthermore, based on field data, some useful specific parametric models for $\rho(t)$ can be developed.

APPENDIX

A. Proof of Theorem 1

Note that the problem is equivalent to the problem of maximizing

$$\Phi(n) \equiv w_1 P(F_1 \mid S) + w_2 P(F_2 \mid W).$$

Substitution gives:

$$\begin{aligned}\Phi(n) &\equiv w_1 P(F_1 \mid S) + w_2 P(F_2 \mid W)\\ &= w_1 P(N(b)\le n \mid S) + w_2 P(N(b) > n \mid W)\\ &= w_1 \sum_{i=0}^{n} \frac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!} + w_2\left(1-\sum_{i=0}^{n}\frac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}\right).\end{aligned}$$

Then observe that, for $n \ge 1$,

$$\Phi(n)-\Phi(n-1) = w_1 \frac{(\Lambda_S(b))^n e^{-\Lambda_S(b)}}{n!} - w_2 \frac{(\Lambda_S(\rho(b)))^n e^{-\Lambda_S(\rho(b))}}{n!} \ge 0$$

$$\Leftrightarrow e^{\Lambda_S(\rho(b))-\Lambda_S(b)} \ge \frac{w_2}{w_1}\left(\frac{\Lambda_S(\rho(b))}{\Lambda_S(b)}\right)^n$$

$$\Leftrightarrow n \le \frac{(\Lambda_S(\rho(b)) - \Lambda_S(b)) + (\ln w_1 - \ln w_2)}{\ln\left(\dfrac{\Lambda_S(\rho(b))}{\Lambda_S(b)}\right)}. \tag{9}$$

Case I. Let

$$\frac{(\Lambda_S(\rho(b)) - \Lambda_S(b)) + (\ln w_1 - \ln w_2)}{\ln\left(\dfrac{\Lambda_S(\rho(b))}{\Lambda_S(b)}\right)} < 1.$$

Then there is no positive integer which satisfies (9). This implies that

$$\Phi(n) - \Phi(n-1) < 0, \forall n \ge 1,$$

and thus we have $n^* = 0$.

Case II. Let

$$\frac{(\Lambda_S(\rho(b)) - \Lambda_S(b)) + (\ln w_1 - \ln w_2)}{\ln\left(\dfrac{\Lambda_S(\rho(b))}{\Lambda_S(b)}\right)} \ge 1.$$

Then n^* is the largest integer which is less than or equal to

$$\frac{(\Lambda_S(\rho(b)) - \Lambda_S(b)) + (\ln w_1 - \ln w_2)}{\ln\left(\dfrac{\Lambda_S(\rho(b))}{\Lambda_S(b)}\right)}.$$

■

B. Proof of Theorem 2

For the fixed $b \ge 0$, we consider the following two cases:

Case 1. Let

$$\left(\Lambda_S(b+\tau) - \Lambda_S(b)\right) \le \left(\Lambda_S(\rho(b+\tau)) - \Lambda_S(\rho(b)\right).$$

As the sum of quotients in equation (11) is 1 in this case, it can be easily seen that minimizing $\Psi(b,n)$ is equivalent to maximizing

$$P(S \mid F_1) = \frac{\left(\sum_{i=0}^{n} \dfrac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}\right) \times p}{\left(\sum_{i=0}^{n} \dfrac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}\right) \times p + \left(\sum_{i=0}^{n} \dfrac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}\right) \times (1-p)}.$$

$$\Leftrightarrow \text{Minimize} \quad \frac{p+(1-p)\times \dfrac{\sum_{i=0}^{n}\dfrac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}}{\sum_{i=0}^{n}\dfrac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}}}{p}$$

$$\Leftrightarrow \text{Minimize} \quad \frac{\sum_{i=0}^{n}\dfrac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}}{\sum_{i=0}^{n}\dfrac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}} \equiv g(b,n)$$

We compare $\Psi(b,n)$ with $\Psi(b,n+1)$, $n=0,1,2,\cdots$. Observe that $g(b,n)<g(b,n+1)$ if and only if $\Psi(b,n)<\Psi(b,n+1)$. Note that

$$\frac{\dfrac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}}{\dfrac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}}$$

is strictly increasing in $i\geq 0$. This can be easily seen by comparing the values of this function for i and $i+1$, $i\geq 0$. Thus

$$\frac{\dfrac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}}{\dfrac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}} < \frac{\dfrac{(\Lambda_S(\rho(b)))^{n+1} e^{-\Lambda_S(\rho(b))}}{(n+1)!}}{\dfrac{(\Lambda_S(b))^{n+1} e^{-\Lambda_S(b)}}{(n+1)!}},\ 0\leq i\leq n.$$

Then using Lemma 1:

$$\frac{\sum_{i=0}^{n}\dfrac{(\Lambda(_S\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}}{\sum_{i=0}^{n}\dfrac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}} < \max_{1\leq i\leq n}\frac{\dfrac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}}{\dfrac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}} < \frac{\dfrac{(\Lambda_S(\rho(b)))^{n+1} e^{-\Lambda_S(\rho(b))}}{(n+1)!}}{\dfrac{(\Lambda_S(b))^{n+1} e^{-\Lambda_S(b)}}{(n+1)!}}.$$

Accordingly, using Lemma 1 again:

$$g(b,n)=\min\left\{\frac{\sum_{i=0}^{n}\dfrac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}}{\sum_{i=0}^{n}\dfrac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}}, \frac{\dfrac{(\Lambda_S(\rho(b)))^{n+1} e^{-\Lambda_S(\rho(b))}}{(n+1)!}}{\dfrac{(\Lambda_S(b))^{n+1} e^{-\Lambda_S(b)}}{(n+1)!}}\right\}$$

$$< \frac{\sum_{i=0}^{n+1}\dfrac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}}{\sum_{i=0}^{n+1}\dfrac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}} = g(b,n+1),$$

implying that $\Psi(b,n)<\Psi(b,n+1)$, $n=0,1,2,\cdots$. Finally, we arrive at $n^*(b)=0$.

This obviously means that for each fixed duration of the burn-in time b, the failed item is discarded and those that did not fail are put into a field operation. Therefore the obtained rule is simple and easy for implementation.

Case 2. Let

$$\left(\Lambda_S(b+\tau)-\Lambda_S(b)\right)>\left(\Lambda_S(\rho(b+\tau))-\Lambda_S(\rho(b)\right).$$

In this case, minimization of $\Psi(b,n)$ is equivalent to minimization of

$$\frac{\left(\sum_{i=0}^{n}\frac{(\Lambda_S(b))^i e^{-\Lambda_S(b)}}{i!}\right)\times p}{\left(\sum_{i=0}^{n}\frac{(\Lambda_S(b))^i e^{-\Lambda(b)}}{i!}\right)\times p+\left(\sum_{i=0}^{n}\frac{(\Lambda_S(\rho(b)))^i e^{-\Lambda_S(\rho(b))}}{i!}\right)\times(1-p)},$$

or, to maximization of $g(b,n)$. Therefore $n^*(b)=\infty$. ■

C. Proof of Theorem 3

Observe that $\Psi(b,n)$ is of the form of weighted average of $\left(\Lambda_S(b+\tau)-\Lambda_S(b)\right)$ and $\left(\Lambda_S(\rho(b+\tau))-\Lambda_S(\rho(b)\right)$, i.e.,

$$\Psi(b,n)=\left(\Lambda_S(b+\tau)-\Lambda_S(b)\right)\times p(b)+\left(\Lambda_S(\rho(b+\tau))-\Lambda_S(\rho(b))\right)\times(1-p(b)),$$

where

$$p(b)=\frac{\left(\sum_{i=0}^{n}\frac{(\Lambda_S(b))^i e^{-\Lambda(b)}}{i!}\right)\times p}{\left(\sum_{i=0}^{n}\frac{(\Lambda_S(b))^i e^{-\Lambda(b)}}{i!}\right)\times p+\left(\sum_{i=0}^{n}\frac{(\Lambda_S(\rho(b)))^i e^{-\Lambda(\rho(b))}}{i!}\right)\times(1-p)}.$$

Also we see that

$$\Lambda_S(b+\tau)-\Lambda_S(b)=\int_b^{b+\tau} r_S(u)du \text{ and } \Lambda_S(\rho(b+\tau))-\Lambda_S(\rho(b))=\int_{\rho(b)}^{\rho(b+\tau)} r_S(u)du.$$

Define $s^*\in[t^*,\infty)$ as

$$s^*=\inf\{b'>t^* \mid \int_{\rho(t^*)}^{\rho(t^*+\tau)} r_S(u)du<\int_b^{b+\tau} r_S(u)du, \forall b>b'\}.$$

It clear that such s^* exists as $\int_b^{b+\tau} r_S(u)du$ is non-decreasing for $b\in[t^*,\infty)$ and is strictly increasing after some point $t'\in[t^*,t^{**}]$. Observe that $\rho(b+\tau)-\rho(b)$ is non-decreasing in b and

$$\Lambda_S(\rho(b+\tau))-\Lambda_S(\rho(b))\geq\Lambda_S(b+\tau)-\Lambda_S(b)$$

for $b\geq t^*$. Then

$$\begin{aligned}\Lambda_S(t^*+\tau)-\Lambda_S(t^*)\leq\Lambda_S(\rho(t^*+\tau))-\Lambda_S(\rho(t^*))<\Lambda_S(b+\tau)-\Lambda_S(b)\\ \leq\Lambda_S(\rho(b+\tau))-\Lambda_S(\rho(b)),\forall b>s^*.\end{aligned}$$

The weighted average of elements in the first group is smaller than that of elements in the second group for any arbitrarily chosen weights in two groups if the maximum element in the first group is smaller than the minimum element in the second group. This fact implies:

$$\Psi(t^*, n) < \Psi(b, n), \forall b > s^*.$$

Then we can conclude that at least the optimal burn-in time $b^* \notin (s^*, \infty)$, i.e., $b^* \leq s^*$. This result holds regardless of the value of n. Therefore, s^* is the uniform (with respect to n) upper bound for b^*. ■

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References

Arjas, E. and Norros, I. (1989). Change of life distribution via a hazard transformation: an inequality with application to minimal repair. *Mathematics of Operations Research*, 14, 355-361.

Barlow, R., Proschan. F. (1975). *Statistical Theory of Reliability and Life Testing.* Holt, Renerhart & Winston, New York.

Block, H. W., Mi, J. and Savits, T. H. (1994). Some results on burn-in, *Statistica Sinica*, 4, 525-533.

Block, H. W. and Savits, T. H. (1997). Burn-in, *Statistical Science*, 12, 1-19.

Block, H. W., Savits, T. H. and Singh, H. (2002). A criterion for burn-in that balances mean residual life and residual variance, *Operations Research*, 50, 290-296.

Block, H. W., Li, Y. and Savits , T. H. (2003a). Initial and final behavior of failure rate functions for mixtures and systems, *Journal of Applied Probability*, 40, 721-740.

Block, H. W., Li, Y. and Savits, T. H. (2003b). Preservation of properties under mixture, Probability in the Engineering and Informational Sciences, 17, 205-212.

Cha, J. H. (2000). On a better burn-in procedure, *Journal of Applied Probability*, 37, 1099-1103.

Cha, J. H. (2006). An extended model for optimal burn-in procedures, *IEEE Transactions on Reliability*, 55, 189-198.

Clarotti, C. A. and Spizzichino, F. (1991). Bayes burn-in decision procedures, *Probability in the Engineering and Informational Sciences*, 4, 437-445.

Finkelstein, M. S.(1992). Some notes on two types of minimal repair. *Advances in Applied Probability*, 24, 226-228.

Finkelstein, M. S. (2004). Minimal repair in heterogeneous populations. *Journal of Applied Probability*, 41, 281-286.

Finkelstein, M. S. (2008), *Failure Rate Modelling for Reliability and Risk*. Springer, London

Gupta, R.C., and Warren, R. (2001). Determination of change points of nonmonotonic failure rates. *Communication in Statistics-Theory and Methods*, 30, 1903-1920.

Gurland, J. and Sethuraman, J. (1995). How pooling failure data may reverse increasing failure rates. *Journal of American Statistical Association*, 90, 1418-1423.

Jensen, F. and Petersen, N. E. (1982). *Burn-In*. Wiley, New York.
Jiang, R., and Murphy, D. N. P. (1995). Modelling failure data by mixture of two Weibull distributions. *IEEE Transactions on Reliability*, 44, 477-488.
Klutke, G., Kiessler, P. C. and Wortman, M. A. (2003). A critical look at the bathtub curve, *IEEE Transactions on Reliability*, 52, 125-129.
Kececioglu, D. and Sun, F. (1995). Environmental Stress Screening : Its Qualification, Optimization, and Management, Prentice Hall.
Kuo, W. and Kuo, Y. (1983). Facing the headaches of early failures: A state-of-the-art review of burn-in decisions. *Proc. IEEE*, 71, 1257-1266.
Mi, J. (1991). *Optimal Burn-In*. Doctoral Thesis, Department of Statistics, Univ. Pittsburgh.
Mi, J. (1994a). Burn-in and maintenance policies, *Advances in Applied Probability*, 26, 207-221.
Mi, J. (1994b). Maximizing of a survival probability and its application, *Journal of Applied Probability*, 31, 1026-1033.
Mi, J. (1996). Minimizing some cost functions related to both burn-in and field use, *Operations Research*, 44, 497-500.
Mi, J. (1997). Warranty policies and burn-in, *Naval Research Logistics*, 44,199-200.
Mi, J. (2002). Age-replacement policy and optimal work size, *Journal of Applied Probability*, 39, 296-311.
Mi, J. (2003). Optimal burn-in time and eventually IFR, *Journal of the Chinese Institute of Industrial Engineers*, 20, 533-542.
Nguyen, D. G. and Murthy, D. N. P. (1982). Optimal burn-in time to minimize cost for products sold under warranty, *IIE Transactions*, 14, 167-174.
Ross, S (1996). *Stochastic Processes*. Wiley, New York.