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**The ages of extremal impact
on life disparity
caused by averting deaths**

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The ages of extremal impact on life disparity caused by averting deaths

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Abstract. This paper is concerned with sensitivity analysis of life disparity with respect to changes in mortality rates. Recently Zhang and Vaupel introduced a "threshold age", such that averting deaths before that age reduces disparity, while averting deaths after that age increases disparity. We provide a refinement to this result by characterizing the ages at which averting deaths has an extremal impact on life disparity. A procedure is given for approaching the threshold age numerically. The results are illustrated using data for the female populations of Denmark, the US, Japan and France in 2005.

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1. Introduction

Following Keyfitz's idea that everybody dies prematurely since every death deprives the person involved of the remainder of his expectation of life [Key77, p.61-68], the measure e^\dagger for the average life expectancy lost due to death, has widely been studied in the literature. It first appeared in [Mit78] and was developed further by Vaupel [Vau86] and recently in [VCR03], [ZV08] and [SAZ⁺09]. Zhang and Vaupel [ZV09] initiated a new direction of analysis, by studying the impact on e^\dagger of a concentrated decrease in mortality at age a .

Life disparity is measured by *life expectancy lost due to death*

$$e^\dagger = \int_0^\infty e(a)d(a)da \quad (1.1)$$

where $e(a)$ is the remaining life expectancy at age a

$$e(a) = \frac{1}{l(a)} \int_a^\infty l(x)dx \quad (1.2)$$

$d(a)$ is the life table distribution of deaths

$$d(a) = l(a)\mu(a) \quad (1.3)$$

$l(a)$ is the probability of survival to age a

$$l(a) = \exp\left(-\int_0^a \mu(x)dx\right) \quad (1.4)$$

and $\mu(a)$ is the age-specific hazard of death. Note that, by equations (1.4) and (1.3),

$$\frac{d}{da}[1 - l(a)] = l(a)\mu(a) = d(a) \quad (1.5)$$

Letting $H(a)$ be the cumulative hazard function

$$H(a) = \int_0^a \mu(x)dx \quad (1.6)$$

in [ZV09], a perturbation is considered that alters $H(a)$ by the step function with a single negative step of size s at age a to get the new cumulative hazard function

$$H_{a,s}(x) = H(x) - s \cdot \mathbb{1}_{[a,\infty)}(x) \quad (1.7)$$

It can be derived, as done in the Appendix, that the corresponding new value of life disparity satisfies

$$e_{a,s}^\dagger = e^\dagger + [\exp(s) - 1]l(a)e(a) \left[H(a) - 1 + \frac{e^\dagger(a)}{e(a)} \right]$$

where $e^\dagger(a)$ is *life expectancy lost due to death among people surviving to age a*

$$e^\dagger(a) = \frac{1}{l(a)} \int_a^\infty e(x) d(x) dx \quad (1.8)$$

Now

$$\frac{\partial}{\partial s} e_{a,s}^\dagger = \exp(s) l(a) e(a) \left[H(a) - 1 + \frac{e^\dagger(a)}{e(a)} \right]$$

and

$$\left. \frac{\partial}{\partial s} e_{a,s}^\dagger \right|_{s=0} = l(a) e(a) \left[H(a) - 1 + \frac{e^\dagger(a)}{e(a)} \right] \quad (1.9)$$

In [ZV09], the function k , where

$$\begin{aligned} k(a) &= \frac{1}{l(a)} \left. \frac{\partial}{\partial s} e_{a,s}^\dagger \right|_{s=0} \\ &= e(a) \left[H(a) - 1 + \frac{e^\dagger(a)}{e(a)} \right] \\ &= e^\dagger(a) + e(a) [H(a) - 1] \end{aligned} \quad (1.10)$$

is analysed and shown to have a positive first derivative at every root, from which it follows that k has at most one root. It is shown that $k(0) < 0$ implies the existence of a unique root a^\dagger (then, of course, positive), $k(a) < 0$ for $a < a^\dagger$ and $k(a) > 0$ for $a > a^\dagger$, and that $k(0) \geq 0$ implies $k(a) > 0$ for $a > 0$.

Since, by equation (1.10), $k(a)$ and $\left. \frac{\partial}{\partial s} e_{a,s}^\dagger \right|_{s=0}$ have the same sign, and since $k(0) = e^\dagger - e(0)$, this is equivalent to saying

Theorem 1.1 *There are the following three cases:*

$$(i) \quad \frac{e^\dagger}{e(0)} < 1 \Rightarrow \exists a^\dagger > 0 :$$

$$\begin{aligned} \forall a < a^\dagger &\Rightarrow \left. \frac{\partial}{\partial s} e_{a,s}^\dagger \right|_{s=0} < 0 \\ a = a^\dagger &\Rightarrow \left. \frac{\partial}{\partial s} e_{a,s}^\dagger \right|_{s=0} = 0 \\ \forall a > a^\dagger &\Rightarrow \left. \frac{\partial}{\partial s} e_{a,s}^\dagger \right|_{s=0} > 0 \end{aligned}$$

$$(ii) \quad \frac{e^\dagger}{e(0)} = 1 \Rightarrow a^\dagger = 0 :$$

$$\begin{aligned} a = a^\dagger &\Rightarrow \left. \frac{\partial}{\partial s} e_{a,s}^\dagger \right|_{s=0} = 0 \\ \forall a > a^\dagger &\Rightarrow \left. \frac{\partial}{\partial s} e_{a,s}^\dagger \right|_{s=0} > 0 \end{aligned}$$

$$(iii) \quad \frac{e^\dagger}{e(0)} > 1 :$$

$$\forall a \Rightarrow \left. \frac{\partial}{\partial s} e_{a,s}^\dagger \right|_{s=0} > 0$$

The unique root a^\dagger of $k(a)$ and $\frac{\partial}{\partial s}e_{a,s}^\dagger|_{s=0}$, if it exists, that is, if $\frac{e^\dagger}{e(0)} \leq 1$, is called **threshold age**.

In this paper we provide a refinement to the above result by characterizing the ages at which averting deaths has an extremal impact (or, in other words, effect) on life disparity. In addition, a procedure is given for approaching the threshold age numerically. The results are illustrated using data for the female populations of Denmark, the US, France and Japan in 2005.

The paper is organised as follows. The main results are presented in Section 2. An explicit example as well as some numerical results using data from the Human Mortality Database are given in Section 3. The proofs of the theorems are collected in Section 4. A formula for the perturbed life disparity is derived in the Appendix.

2. Main results

The function $\frac{\partial}{\partial s}e_{a,s}^\dagger|_{s=0}$ (cf. (1.9)) measures the effect on life disparity of averting deaths at age a . It turns out that looking at this function directly allows us to make interesting statements about its behaviour, beyond those about the sign obtained by looking at k (cf. (1.10)) and summarised in Theorem 1.1.

2.1. The effect on life disparity caused by averting deaths

The main result of this paper is the following

Theorem 2.1 *The function φ , defined by*

$$\varphi(a) = \frac{\partial}{\partial s}e_{a,s}^\dagger \Big|_{s=0} \tag{2.1}$$

has the following properties.

(i) *Let \tilde{a} , the age of cumulative hazard unity, be defined via*

$$H(\tilde{a}) = 1$$

Then φ is strictly increasing on $[0, \tilde{a}]$ and strictly decreasing and strictly positive on $[\tilde{a}, \infty)$, having

- a global maximum of $\varphi(\tilde{a}) = l(\tilde{a})e^\dagger(\tilde{a})$ at $a = \tilde{a}$ and*
- a local minimum of $\varphi(0) = e^\dagger(0) - e(0)$ at $a = 0$.*

More precisely,

$$\frac{d}{da}\varphi(a) = l(a)[1 - H(a)]$$

(ii) If, for some $\varepsilon > 0$ and $A \geq 0$, we have

$$\mu(a) \geq \varepsilon \quad \forall a \geq A$$

then

$$\lim_{a \rightarrow \infty} \varphi(a) = 0$$

(iii) Let a^* be defined by

$$H(a^*) = 2$$

Then φ is strictly concave on $[0, a^*]$ and strictly convex on $[a^*, \infty)$. More precisely,

$$\frac{d^2}{da^2} \varphi(a) = d(a)[H(a) - 2]$$

Regarding the existence of \tilde{a} and a^* , note that, by equation (1.6), H is strictly increasing with $H(0) = 0$, and that $\lim_{a \rightarrow \infty} H(a) = \infty$ is generally assumed.

Theorem 2.1(i) implies that φ has at most one root, which exists if and only if $\varphi(0) \leq 0 \iff e^\dagger(0) - e(0) \leq 0 \iff \frac{e^\dagger}{e(0)} \leq 1$, so that Theorem 2.1(i) implies Theorem 1.1. Refining Theorem 1.1, our Theorem 2.1 highlights some helpful monotonicity properties and draws attention to \tilde{a} and 0, the ages of extremal effect on life disparity caused by averting deaths.

Consider ages $a > a^\dagger$, if a^\dagger exists, and consider any age otherwise. By Theorem 1.1, life disparity increases by averting deaths at any such age. More precisely, by Theorem 2.1(i), **the increase in life disparity by averting deaths is the higher the closer the age is to \tilde{a}** , where the increase is highest.

Consider ages $a < a^\dagger$, if a^\dagger exists. By Theorem 1.1, life disparity decreases by averting deaths at any such age. More precisely, by Theorem 2.1(i), **the decrease in life disparity by averting deaths is the higher the closer the age is to 0**, where the decrease is highest.

2.2. Approaching the threshold age numerically

According to Theorem 2.1(iii), since $\tilde{a} < a^*$ clearly holds, φ is strictly concave on $[0, \tilde{a}]$. This is the property we can make use of for approximating a^\dagger , provided such an approximation makes sense, that is, $\varphi(0) < 0$.

First, we can construct a strictly increasing sequence that approaches a^\dagger from below. Second, we can construct a strictly decreasing sequence that approaches a^\dagger from above.

Theorem 2.2 Assume $\frac{e^\dagger}{e(0)} < 1$ and let a^\dagger be the threshold age, defined by $\varphi(a^\dagger) = 0$.

(i) The sequence (x_n) , defined by

$$x_{n+1} = x_n + e(x_n) - \frac{e^\dagger(x_n)}{1 - H(x_n)}, \quad x_0 = 0$$

strictly increases and rapidly (quadratically) converges to a^\dagger . In particular,

$$x_1 = e(0) - e^\dagger(0)$$

is a first non-trivial approximation of a^\dagger from below.

(ii) The sequence (y_n) , defined by

$$y_{n+1} = y_n \cdot \frac{e(0) - e^\dagger(0)}{l(y_n)e(y_n)[H(y_n) - 1] + l(y_n)e^\dagger(y_n) + e(0) - e^\dagger(0)}, \quad y_0 = \tilde{a}$$

strictly decreases and (linearly) converges to a^\dagger . In particular,

$$y_1 = \tilde{a} \cdot \frac{e(0) - e^\dagger(0)}{l(\tilde{a})e^\dagger(\tilde{a}) + e(0) - e^\dagger(0)}$$

is a first non-trivial approximation of a^\dagger from above.

Concluding for x_1 and y_1 of Theorem 2.2, we have $a^\dagger \in (x_1, y_1) \subset [0, \tilde{a}]$. This, of course, does not locate a^\dagger precisely, but constitutes **new non-trivial bounds on a^\dagger** . First, we can say that **life disparity decreases by averting deaths at any age $a \leq x_1 = e(0) - e^\dagger(0)$** (with the decrease being the higher the closer the age is to 0). Second, we can say that **life disparity increases by averting deaths at any age $a \geq y_1 = \tilde{a} \cdot \frac{e(0) - e^\dagger(0)}{l(\tilde{a})e^\dagger(\tilde{a}) + e(0) - e^\dagger(0)}$** (with the increase being the higher the closer the age is to \tilde{a}).

3. Illustrations

3.1. Constant hazards

Constant hazards are unrealistic for human populations, where, roughly speaking, between ages 30 and 90 the hazard is exponential. However, hypothetical populations with constant hazards of death can be used to illustrate Theorem 2.1. The effect on life disparity caused by averting deaths at age a should be strictly increasing from age 0 to \tilde{a} and strictly decreasing (and strictly positive) from age \tilde{a} onwards.

Indeed, suppose

$$\mu(a) = C$$

By equation (1.6),

$$H(a) = \int_0^a \mu(x)dx = Ca \tag{3.1}$$

By equations (1.4) and (1.6),

$$l(a) = \exp(-Ca) \quad (3.2)$$

By equation (1.3),

$$d(a) = l(a)\mu(a) = C \exp(-Ca)$$

By equation (1.2),

$$\begin{aligned} e(a) &= \frac{1}{l(a)} \int_a^\infty l(x) dx \\ &= \exp(Ca) \int_a^\infty \exp(-Cx) dx \\ &= \exp(Ca) \left[\frac{-1}{C} \exp(-Cx) \right]_a^\infty \\ &= \exp(Ca) \left[\frac{1}{C} \exp(-Ca) \right] = \frac{1}{C} \end{aligned} \quad (3.3)$$

By equations (1.8) and (1.5),

$$\begin{aligned} e^\dagger(a) &= \frac{1}{l(a)} \int_a^\infty e(x) d(x) dx \\ &= \frac{1}{Cl(a)} [1 - l(x)]_a^\infty = \frac{1}{C} \end{aligned} \quad (3.4)$$

Finally, by equations (4.2), (4.1), (3.2), (3.3), (3.1) and (3.4),

$$\begin{aligned} \varphi(a) &= l(a)e(a)\psi(a) \\ &= l(a)e(a) \left[H(a) - 1 + \frac{e^\dagger(a)}{e(a)} \right] \\ &= \exp(-Ca) \frac{1}{C} \left[Ca - 1 + \frac{1}{C} \right] \\ &= a \exp(-Ca) \end{aligned}$$

Clearly, $a^\dagger = 0$. Since \tilde{a} is defined by $1 = H(\tilde{a}) = C\tilde{a}$, we have $\tilde{a} = \frac{1}{C}$. It is straightforward to see that φ is indeed strictly increasing from age 0 to \tilde{a} and strictly decreasing (and strictly positive) from age \tilde{a} onwards.

3.2. Numerical findings

To illustrate the theoretical results, we have computed several relevant quantities for four life tables from the Human Mortality Database 2010 [Dat10]. In the figures below the function φ , representing the effect on life disparity caused by averting deaths (see Theorem 2.1), is displayed by a solid line. The approximations x_1 and y_1 of the threshold age (see Theorem 2.2) are obtained by the intersections of the dashed lines and the age axis.

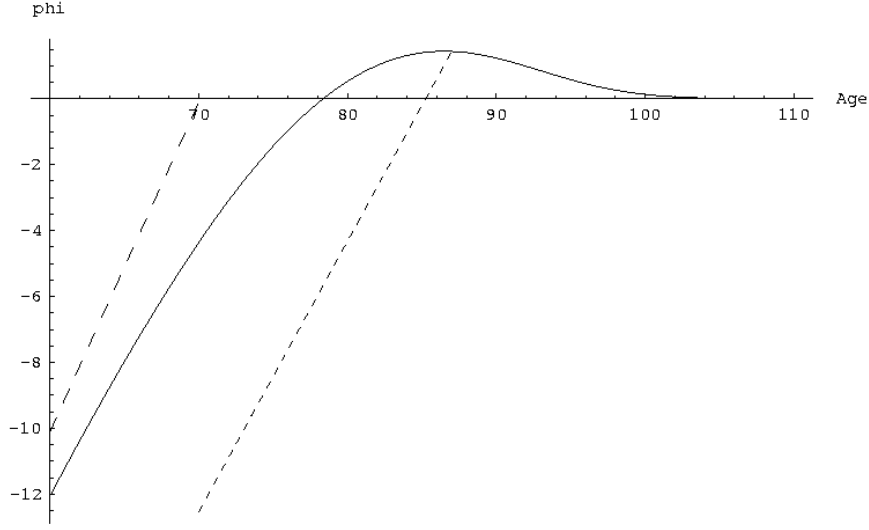


Figure 1: The effect on life disparity caused by averting deaths for Danish females in 2005

For females in Denmark in 2005, the age of cumulative hazard unity \tilde{a} is about 87 years, and x_1 , a^\dagger and y_1 are approximately 70, 78 and 85 years, respectively, as shown in Figure 1. For US-females in 2005, \tilde{a} is around 87 years, and x_1 , a^\dagger and y_1 are approximately 69, 79 and 85 years, respectively, as shown in Figure 2. For females in France in 2005, \tilde{a} is about 90 years, and x_1 , a^\dagger and y_1 are approximately 74, 83 and 88 years, respectively, as shown in Figure 3. Finally, for Japanese females in 2005, \tilde{a} is around 91 years, and x_1 , a^\dagger and y_1 are approximately 76, 84 and 89 years, respectively, as shown in Figure 4. Note that the four obtained threshold ages concur with Figure 2 of [ZV09]. Note further that our Figure 2 concurs with Figure 1 of [ZV09].

It is evident that in all four cases the approximations x_1 and y_1 of a^\dagger lie within ten years of a^\dagger , and the arithmetic mean of x_1 and y_1 lies within two years of a^\dagger . In fact, \tilde{a} also lies within ten years of a^\dagger , and the arithmetic mean of x_1 and \tilde{a} even lies within one year of a^\dagger . Whether the arithmetic mean of x_1 and \tilde{a} is, in general, a better approximation of a^\dagger than the arithmetic mean of x_1 and y_1 , would be an interesting question for future research. If so, it would emphasise the importance of \tilde{a} , the age of cumulative hazard unity, and, due to the simple definitions of x_1 and \tilde{a} , improve our understanding of a^\dagger .

It should be mentioned that in all four cases, the extremal effects on life disparity caused by averting deaths are far from being equal. Indeed, for Denmark, $\varphi(0) = -x_1 \approx -70.12$ and $\varphi(87) \approx 1.44$ with $-\varphi(0)/\varphi(87) \approx 48.7$, for the US, $\varphi(0) = -x_1 \approx -68.87$ and $\varphi(87) \approx 1.56$ with $-\varphi(0)/\varphi(87) \approx 44.1$, for France, $\varphi(0) = -x_1 \approx -74.08$ and $\varphi(90) \approx 1.29$ with $-\varphi(0)/\varphi(90) \approx 57.4$, and

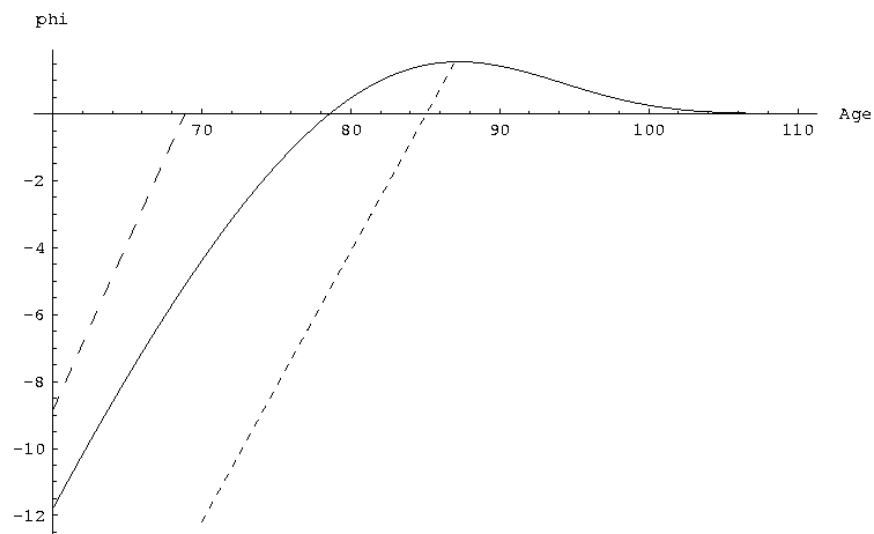


Figure 2: The effect on life disparity caused by averting deaths for US-females in 2005

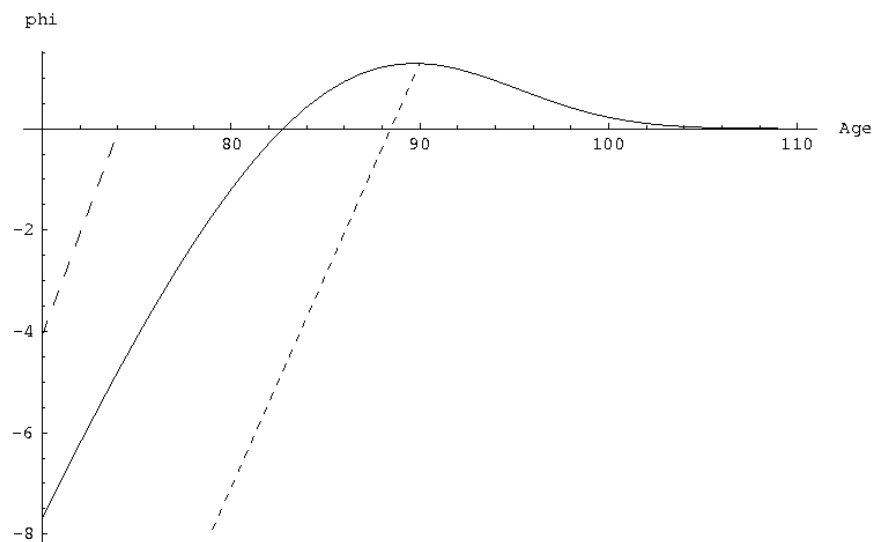


Figure 3: The effect on life disparity caused by averting deaths for French females in 2005

for Japan, $\varphi(0) = -x_1 \approx -75.88$ and $\varphi(91) \approx 1.34$ with $-\varphi(0)/\varphi(91) \approx 56.6$, so that, in absolute value, the effect on life disparity by averting deaths at age zero is roughly 50 times larger than the effect on life disparity by averting deaths at age \tilde{a} . Studying this ratio for other life tables and possible implications might be another interesting direction of future research.

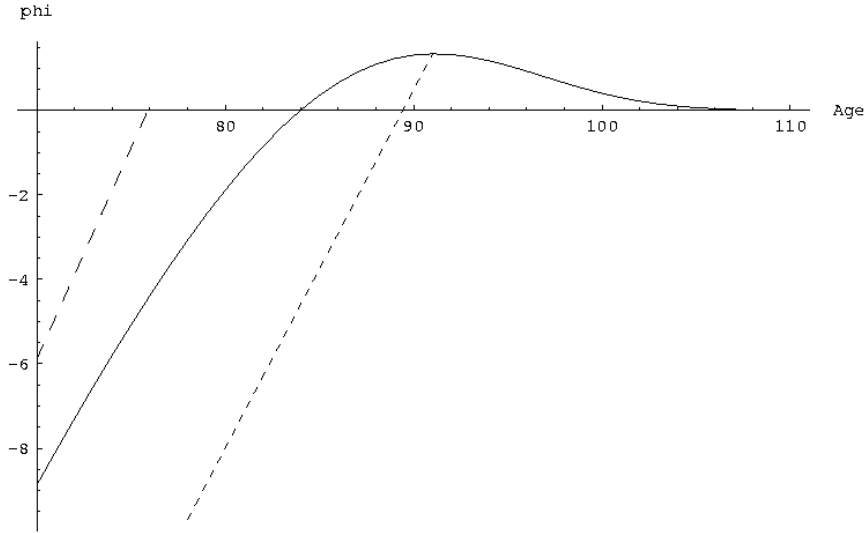


Figure 4: The effect on life disparity caused by averting deaths for Japanese females in 2005

4. Proofs

4.1. Proof of Theorem 2.1

For proving Theorem 2.1, we need the following result.

Lemma 4.1 *The function ψ , defined by*

$$\psi(a) = H(a) - 1 + \frac{e^\dagger(a)}{e(a)} \quad (4.1)$$

is strictly increasing. More precisely,

$$\frac{d}{da}\psi(a) = \frac{e^\dagger(a)}{e(a)^2} > 0$$

Proof. By equations (1.6), (1.2), (1.8) and (1.3),

$$\begin{aligned}
\frac{d}{da}\psi(a) &= \frac{d}{da}H(a) + \frac{d}{da}\frac{e^\dagger(a)}{e(a)} \\
&= \frac{d}{da}\int_0^a \mu(x)dx + \frac{d}{da}\frac{l(a)e^\dagger(a)}{l(a)e(a)} \\
&= \mu(a) + \frac{\left[\frac{d}{da}\int_a^\infty e(x)d(x)dx\right]l(a)e(a) - l(a)e^\dagger(a)\left[\frac{d}{da}\int_a^\infty l(x)dx\right]}{[l(a)e(a)]^2} \\
&= \mu(a) + \frac{[-e(a)d(a)]l(a)e(a) - l(a)e^\dagger(a)[-l(a)]}{[l(a)e(a)]^2} \\
&= \mu(a) - \frac{d(a)}{l(a)} + \frac{e^\dagger(a)}{e(a)^2} \\
&= \frac{e^\dagger(a)}{e(a)^2} > 0
\end{aligned}$$

showing that ψ is a strictly increasing function.

Proof of part (i)

By equations (1.9), (2.1) and (4.1),

$$\varphi(a) = l(a)e(a)\psi(a) \quad (4.2)$$

First, by Lemma 4.1 and equation (4.1), we have $\psi(a) > \psi(\tilde{a}) = \frac{e^\dagger(\tilde{a})}{e(\tilde{a})} > 0$, for all $a > \tilde{a}$. Consequently, by equations (4.2) and (1.2), φ is strictly positive on $[\tilde{a}, \infty)$.

Then, by equations (4.2), (1.2), (4.1) and Lemma 4.1, we have

$$\begin{aligned}
\frac{d}{da}\varphi(a) &= \frac{d}{da}[l(a)e(a)\psi(a)] \\
&= \left\{\frac{d}{da}[l(a)e(a)]\right\}\psi(a) + [l(a)e(a)]\frac{d}{da}\psi(a) \\
&= \left[\frac{d}{da}\int_a^\infty l(x)dx\right]\left[H(a) - 1 + \frac{e^\dagger(a)}{e(a)}\right] + [l(a)e(a)]\frac{e^\dagger(a)}{e(a)^2} \\
&= [-l(a)]\left[H(a) - 1 + \frac{e^\dagger(a)}{e(a)}\right] + l(a)\frac{e^\dagger(a)}{e(a)} \\
&= l(a)[1 - H(a)] \quad (4.3)
\end{aligned}$$

So the first derivative of φ is strictly positive on $[0, \tilde{a})$ and strictly negative on (\tilde{a}, ∞) . Thus, φ is strictly increasing on $[0, \tilde{a}]$ and strictly decreasing on $[\tilde{a}, \infty)$. Consequently, φ has a global maximum at $a = \tilde{a}$, where its value, by equations (4.2) and (4.1), is

$$\varphi(\tilde{a}) = l(\tilde{a})e^\dagger(\tilde{a})$$

and a local minimum at $a = 0$, where its value, by equations (4.2) and (4.1), is

$$\varphi(0) = e^\dagger(0) - e(0) \quad (4.4)$$

Proof of part (ii)

By equations (4.2), (4.1), (1.2), (1.8), (1.4) and (1.6),

$$\begin{aligned}
\varphi(a) &= l(a)e(a)\psi(a) \\
&= l(a)e(a) \left[H(a) - 1 + \frac{e^\dagger(a)}{e(a)} \right] \\
&= H(a) \int_a^\infty l(x)dx - \int_a^\infty l(x)dx + \int_a^\infty e(x)d(x)dx \\
&= H(a) \int_a^\infty \exp[-H(x)]dx - \int_a^\infty l(x)dx + \int_a^\infty e(x)d(x)dx
\end{aligned}$$

Clearly, $\lim_{a \rightarrow \infty} \int_a^\infty l(x)dx = 0$ and $\lim_{a \rightarrow \infty} \int_a^\infty e(x)d(x)dx = 0$, so that it remains to show $\lim_{a \rightarrow \infty} H(a) \int_a^\infty \exp[-H(x)]dx = 0$.

Let $c = \frac{\varepsilon}{2}$. Since $\mu(a) \geq \varepsilon$ for all $a \geq A$, we have

$$[H(x) - cx] - [H(a) - ca] = \int_a^x [\mu(y) - c]dy \geq 0 \quad \forall x \geq a \geq A \quad (4.5)$$

and

$$H(a) - ca \geq \int_A^a \mu(x)dx - ca \geq ca - 2cA \quad \forall a \geq A \quad (4.6)$$

It follows from equation (4.5) that

$$\begin{aligned}
\int_a^\infty \exp[-H(x)]dx &= \int_a^\infty \exp\{ -[H(x) - cx] - cx \}dx \\
&\leq \exp\{ -[H(a) - ca] \} \int_0^\infty \exp(-cx)dx \\
&= \frac{1}{c} \exp\{ -[H(a) - ca] \} \quad \forall a \geq A \quad (4.7)
\end{aligned}$$

One obtains further with equations (4.7) and (4.6) that

$$\begin{aligned}
H(a) \int_a^\infty \exp[-H(x)]dx &\leq \frac{1}{c} H(a) \exp\{ -[H(a) - ca] \} \\
&\leq \frac{1}{c} [H(a) - ca] \exp\{ -[H(a) - ca] \} + \\
&\quad a \exp[-(ca - 2cA)] \quad \forall a \geq A
\end{aligned}$$

By equation (4.6), $\lim_{a \rightarrow \infty} \frac{1}{c} [H(a) - ca] \exp\{ -[H(a) - ca] \} = 0$, and clearly $\lim_{a \rightarrow \infty} a \exp[-(ca - 2cA)] = 0$. Hence $\lim_{a \rightarrow \infty} H(a) \int_a^\infty \exp[-H(x)]dx = 0$, which finishes the proof.

Proof of part (iii)

By equations (4.3), (1.5), (1.6) and (1.3),

$$\frac{d}{da} \left[\frac{d}{da} \varphi(a) \right] = \frac{d}{da} \{ l(a)[1 - H(a)] \}$$

$$\begin{aligned}
&= \left[\frac{d}{da} l(a) \right] [1 - H(a)] + l(a) \frac{d}{da} [1 - H(a)] \\
&= [-d(a)] [1 - H(a)] + l(a) [-\mu(a)] \\
&= d(a) [H(a) - 1] - l(a) \mu(a) \\
&= d(a) [H(a) - 2]
\end{aligned}$$

So the second derivative of φ is strictly negative on $[0, a^*)$ and strictly positive on (a^*, ∞) . Thus, φ is strictly concave on $[0, a^*]$ and strictly convex on $[a^*, \infty)$.

4.2. Proof of Theorem 2.2

Note that $a^\dagger < \tilde{a}$ and, by Theorem 2.1(iii), φ is strictly concave on $[0, \tilde{a}]$.

By Newton's method, the sequence (x_n) , defined by

$$x_{n+1} = x_n - \frac{\varphi(x_n)}{\frac{d}{da}\varphi(x_n)}, \quad x_0 = 0$$

strictly increases and rapidly (quadratically) converges to a^\dagger . In our particular case, by equations (4.2) and (4.1) and Theorem 2.1(i),

$$\begin{aligned}
x_{n+1} &= x_n - \frac{l(x_n)e(x_n)\psi(x_n)}{l(x_n)[1 - H(x_n)]} \\
&= x_n - \frac{e(x_n) \left[H(x_n) - 1 + \frac{e^\dagger(x_n)}{e(x_n)} \right]}{1 - H(x_n)} \\
&= x_n + e(x_n) - \frac{e^\dagger(x_n)}{1 - H(x_n)}
\end{aligned}$$

By the Regula Falsi method, the sequence (y_n) , defined by

$$y_{n+1} = y_n \cdot \frac{-\varphi(0)}{\varphi(y_n) - \varphi(0)}, \quad y_0 = \tilde{a}$$

strictly decreases and (linearly) converges to a^\dagger . In our particular case, by equations (4.4), (4.2) and (4.1),

$$\begin{aligned}
y_{n+1} &= y_n \cdot \frac{-[e^\dagger(0) - e(0)]}{l(y_n)e(y_n)\psi(y_n) - [e^\dagger(0) - e(0)]} \\
&= y_n \cdot \frac{e(0) - e^\dagger(0)}{l(y_n)e(y_n) \left[H(y_n) - 1 + \frac{e^\dagger(y_n)}{e(y_n)} \right] + e(0) - e^\dagger(0)} \\
&= y_n \cdot \frac{e(0) - e^\dagger(0)}{l(y_n)e(y_n)[H(y_n) - 1] + l(y_n)e^\dagger(y_n) + e(0) - e^\dagger(0)}
\end{aligned}$$

5. Appendix: Calculating the perturbed life disparity

Suppose that the cumulative hazard function H gets replaced by $H_{a,s}$, defined in equation (1.7). According to equation (1.6), μ then gets replaced by $\mu_{a,s}$, where

$$\mu_{a,s} = \mu - s \cdot \delta_a$$

According to equations (1.4) and (1.6), l then gets replaced by $l_{a,s}$, where

$$l_{a,s} = l \cdot \exp\{s \cdot \mathbb{1}_{[a,\infty)}\} \quad (5.1)$$

According to equation (1.5), it can be shown that d then gets replaced by $d_{a,s}$, where

$$d_{a,s} = \{1 + [\exp(s) - 1] \cdot \mathbb{1}_{[a,\infty)}\} \cdot d + [1 - \exp(s)]l(a) \cdot \delta_a \quad (5.2)$$

Indeed, for $x < a$, we have $\int_0^x d_{a,s}(t)dt = \int_0^x d(t)dt = 1 - l(x) = 1 - l_{a,s}(x)$. For $x \geq a$, we have $\int_0^x d_{a,s}(t)dt = [1 - \exp(s)]l(a) + \int_0^a d(t)dt + \int_a^x \exp(s)d(t)dt$, where the right hand side simplifies to $[1 - \exp(s)]l(a) + [1 - l(a)] + \exp(s)[l(a) - l(x)] = 1 - \exp(s)l(x) = 1 - l_{a,s}(x)$.

According to equation (1.2), e then gets replaced by $e_{a,s}$, where

$$e_{a,s}(x) = \frac{1}{l_{a,s}(x)} \int_x^\infty l_{a,s}(y)dy$$

By equation (5.1), we obtain

$$e_{a,s}(x) = \frac{1}{(l \cdot \exp\{s \cdot \mathbb{1}_{[a,\infty)}\})(x)} \int_x^\infty (l \cdot \exp\{s \cdot \mathbb{1}_{[a,\infty)}\})(y)dy \quad (5.3)$$

For $x < a$, we thus have

$$\begin{aligned} e_{a,s}(x) &= \frac{1}{l(x)} \int_x^\infty (l \cdot \exp\{s \cdot \mathbb{1}_{[a,\infty)}\})(y)dy \\ &= \frac{1}{l(x)} \left[\int_x^a l(y)dy + \int_a^\infty \exp(s)l(y)dy \right] \\ &= \frac{1}{l(x)} \left[\int_x^\infty l(y)dy - \int_a^\infty l(y)dy + \exp(s) \int_a^\infty l(y)dy \right] \\ &= \frac{1}{l(x)} [l(x)e(x) - l(a)e(a) + \exp(s)l(a)e(a)] \\ &= e(x) + [\exp(s) - 1] \frac{l(a)}{l(x)} e(a) \end{aligned} \quad (5.4)$$

Note that if $x < a$ is very close to a , then $e_{a,s}(x)$ is approximately $\exp(s)e(a)$. For $x \geq a$, by equation (5.3), we have

$$e_{a,s}(x) = \frac{1}{\exp(s)l(x)} \int_x^\infty \exp(s)l(y)dy = \frac{1}{l(x)} \int_x^\infty l(y)dy = e(x) \quad (5.5)$$

Hence, by combining equations (5.4) and (5.5),

$$e_{a,s}(x) = e(x) + [\exp(s) - 1]l(a)e(a) \cdot \left\{ \frac{\mathbb{1}_{[0,a)}}{l} \right\}(x) \quad (5.6)$$

Finally, according to equation (1.1), e^\dagger then gets replaced by $e_{a,s}^\dagger$, where

$$e_{a,s}^\dagger = \int_0^\infty e_{a,s}(x) d_{a,s}(x) dx$$

By equations (5.2) and (5.6), we obtain

$$\begin{aligned} e_{a,s}^\dagger &= \int_0^\infty e(x) \{1 + [\exp(s) - 1]\mathbb{1}_{[a,\infty)}(x)\} d(x) dx + \\ &\quad \int_0^\infty e(x) [1 - \exp(s)] l(a) \delta_a(x) dx + \\ &\quad \int_0^\infty [\exp(s) - 1] l(a) e(a) \frac{\mathbb{1}_{[0,a)}(x)}{l(x)} \{1 + [\exp(s) - 1]\mathbb{1}_{[a,\infty)}(x)\} d(x) dx + \\ &\quad \int_0^\infty [\exp(s) - 1] l(a) e(a) \frac{\mathbb{1}_{[0,a)}(x)}{l(x)} [1 - \exp(s)] l(a) \delta_a(x) dx \\ &= \left(\int_0^\infty e(x) d(x) dx + [\exp(s) - 1] \int_a^\infty e(x) d(x) dx \right) + \\ &\quad e(a) [1 - \exp(s)] l(a) + \\ &\quad [\exp(s) - 1] l(a) e(a) \int_0^a \frac{d(x)}{l(x)} dx + \\ &\quad 0 \end{aligned}$$

Furthermore, by using equations (1.1), (1.3), (1.6) and (1.8), we obtain

$$\begin{aligned} e_{a,s}^\dagger &= e^\dagger + [\exp(s) - 1] l(a) e^\dagger(a) + [1 - \exp(s)] l(a) e(a) + \\ &\quad [\exp(s) - 1] l(a) e(a) H(a) \\ &= e^\dagger + [\exp(s) - 1] l(a) e(a) \left[H(a) - 1 + \frac{e^\dagger(a)}{e(a)} \right] \end{aligned}$$

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