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Abstract. In this paper the Strehler-Mildvan theory of mortality and aging is generalised to cover heterogeneity effects in the population. The theory is based on the concept of environmental shocks that cause death of an individual when exceeding its vitality. Heterogeneity is introduced via the value of the vitality of an individual at birth. The main result of the paper is an expression for the observed mortality rate of the heterogeneous population. This mortality rate grows according to Gompertz's Law at midlife-ages, then its growth declines, levelling off at high ages. This behaviour is qualitatively consistent with real mortality rates, which is illustrated for period data of Japanese females in the years 1947, 1967, 1987 and 2007. Finally, the duality between a continuous SM-version and the $\varrho-\gamma-$Gompertz model is discussed.

Contents

1. Introduction

In 1960 Strehler and Mildvan [SM60] published a general theory of mortality and aging based on environmental shocks as the key cause of death. They assume homogeneousness for populations and only discuss heterogeneity aspects in the penultimate section (cf. [SM60], page 19). Strehler and Mildvan deduce a linear decline in *vitality* from the Gompertz-mortality function. "Vitality is defined as the capacity of an individual organism to stay alive" and can be regarded as the difference of the total and the basal power output of an organism. The linear decline in vitality is supported by data on many physiological functions (see, for example, Shock (1957) [Sho57]).

An overview of further developments is presented by Yashin, Iachine and Begun (2000) [YIB00]. Sacher and Trucco (1962) [ST62] define some quantity similar to vitality, which they assume to decline linearly. Based on a set of further assumptions, they deduce the Gompertz-hazard. While Strehler and Mildvan regard environmental shocks in terms of energy, Atlan [Atl68] studies those challenges in terms of entropy. Yashin, Begun, Boiko, Ukraintseva and Oeppen (2001) [YBB$^+$01] focus on the so-called SM correlation and conclude that "the instability of SM correlation requires further extension of the SM theory", suggesting that "continuous changes in all parameters of the SM model have to be assumed". Finkelstein (2007) [Fin07] thoroughly revisits the SM theory and outlines some shortcomings of [SM60].

Zheng, Yang and Land (2011) [ZYL11] discuss heterogeneity in the Strehler-Mildvan model. They repeatedly use the age of expected zero vitality and the expected maximum survival age synonymously, which is also mentioned in [SM60], where it is claimed "that the theory predicts that the maximum lifetime attainable in a homogeneous population will be approximately $1/B$", where "B is the fractional loss each year of original vitality". We outline some deficiencies of [SM60] and propose a refined and simple theory of mortality. According to our theory, death is not certain at the age of zero vitality. When introducing the extensive studies in [ZYL11], it is concluded that "these findings indicate that the SM theory needs to be generalized to incorporate heterogeneity among human populations". We introduce heterogeneity via the value of the vitality of an individual at birth. The main result of the paper is an expression for the observed mortality rate of the heterogeneous population. This mortality rate grows according to Gompertz's Law up to an inflection point, then its growth declines, reaching a constant plateau at the end. This behaviour is qualitatively consistent with real mortality rates, which is illustrated for period data of Japanese females in the years 1947, 1967, 1987 and 2007. Our theory thus supports the findings of Robine, Cournil, Gampe and Vaupel (2005) [RCGV05] on constant mortality hazards from the age of 110 years to the age of 114 years, with annual probabilities of death close to 0.5. Finally, the duality between a continuous SM-version and the $\varrho-\gamma-$Gompertz model is discussed.

2. Results

2.1. “Standard” SMT

The main assumption of the SMT is the following **Postulate (1)**.

(1) The sole cause of death for an organism is an environmental challenge whose magnitude M exceeds the current vitality V of the organism.

  (1a) The waiting times ξ_i between challenges are independent and exponentially distributed with parameter λ, that is, $\mathbb{P}(\,\xi_i \geq s\,) = \exp(-\lambda s)$, for all $s \geq 0$.

  (1b) The magnitudes M_i of the challenges are independent of each other (and of the ξ_i) and have the same continuous probability distribution.

  (1c) The vitality $V(x)$ of an organism is a continuous non-negative function of its age x.

Let the force of mortality $\mu(x)$ at age x be defined as usual, that is, as the limit of the ratio of the conditional probability of death in the age interval $[x, x+\varepsilon]$ given survival until age x and the width $\varepsilon > 0$ of the interval, as ε tends to zero. Then Theorem 1 expresses $\mu(x)$ via the parameters from Postulate (1).

Theorem 1 *Postulate (1) implies that the force of mortality $\mu(x)$ satisfies*

$$\mu(x) \;=\; \lambda\, \mathbb{P}[M \geq V(x)]\,. \tag{2.1}$$

The next two postulates provide specifications of the parameters M and V that occur in Postulate (1).

Postulate (2) concerns the environmental parameter M.

(2) The magnitudes M_i of the challenges are exponentially distributed with parameter ν.

This has the following consequence for the mortality rate.

Corollary 1 *It follows from Theorem 1 and Postulate (2) that the force of mortality $\mu(x)$ satisfies*

$$\mu(x) \;=\; \lambda\, \exp[-\nu V(x)]\,. \tag{2.2}$$

Postulate (3) concerns the individual parameter V.

(3) The vitality satisfies

$$V(x) = \begin{cases} V_0\,(1 - Bx) & \text{if} \quad 0 \leq x \leq \frac{1}{B}\,, \\[2ex] 0 & \text{if} \quad x \geq \frac{1}{B}\,, \end{cases} \tag{2.3}$$

where $V_0 > 0$ and $B > 0$ are some constants.

This has the following consequence for the mortality rate.

Corollary 2 *It follows from Corollary 1 and Postulate (3) that the force of mortality $\mu(x)$ satisfies*

$$\mu(x) = \begin{cases} a\ \exp(bx) & \text{if} \quad 0 \le x \le \frac{1}{B}\,, \\ \lambda & \text{if} \quad x \ge \frac{1}{B}\,, \end{cases} \tag{2.4}$$

where

$$a = \lambda\ \exp(-\nu V_0) \tag{2.5}$$

and

$$b = \nu V_0 B\,. \tag{2.6}$$

Discussion of the theory

- Postulate (1) describes a random mechanism governing the occurrence of death of an organism. The stochastic process is composed of an environmental and an individual component. The environment is described by the shock frequency and the distribution of the magnitude of shocks. The individual is characterised by an age-dependent vitality.

- Theorem 1 shows that the force of mortality is always bounded above by the parameter characterising the frequency of shocks. This holds for arbitrary distributions of the shock magnitudes and arbitrary forms of the vitality function. In particular, this shows that **no certain death** occurs, even for zero vitality.

- Corollary 1 shows the impact of Postulate (2) concerning the distribution of shock magnitudes on the expression of the force of mortality. Postulate (2) is related to the Maxwell-Boltzmann energy distribution.

- Corollary 2 shows the impact of Postulate (3) concerning the form of the vitality function on the expression of the force of mortality. As long as the vitality is linearly decreasing and positive, the force of mortality is Gompertz-like. However, when the vitality is zero, the force of mortality equals the parameter characterising the frequency of shocks.

- Finally, Corollary 2 entails an interesting by-product: The well-known “inverse relationship” or “negative correlation” between the Gompertz-parameters, also called “SM correlation” (see, for example, [YBB+01]), follows directly from varying the parameter characterising the magnitude of shocks in (2.5) and (2.6), while keeping the other parameters fixed.

2.2. Heterogeneity (and time-dependence)

Corollary 2 shows that an individual's force of mortality is exponentially increasing as a Gompertz-curve until hitting a ceiling at the age of zero vitality. However, instead of a sharp change from exponential increase to a constant plateau, we can only observe a gradual levelling off at high ages (see, for example, [RCGV05]) at the population level. This may be explained with heterogeneity among individuals. Furthermore, observed mortality curves have changed significantly in recent history, so that including a time-dependency in our model appears to be appropriate.

In the standard SMT, there are two parameters (V_0 and B) forming the individual component and two parameters (λ and ν) forming the environmental component. In the following we introduce new relationships between them and heterogeneity and time. More precisely, for simplicity, we relate the individual component and heterogeneity, and the environmental component and time.

Individual parameters

Equation (2.3) defines the vitality as a linear function of age with intercept V_0 and slope $-V_0 B$ on the interval $\left[0, \frac{1}{B}\right]$ and zero otherwise. In line with a well-known hypothesis supporting a constant intrinsic (with disregard to environmental influences) rate of aging across all individuals, (2.6) suggests to keep the slope $-V_0 B$ constant and, consequently, to equip the intercept V_0 with all heterogeneity. Calling the slope $-\beta$, that is,

$$\beta = V_0 B\,, \tag{2.7}$$

and the intercept z, that is,

$$z = V_0\,, \tag{2.8}$$

where z has the density function $p(x, y, z)$ at age x and time y, (2.3) becomes

$$V(x,z) = \begin{cases} z - \beta x & \text{if} \quad 0 \le x \le \frac{z}{\beta}\,, \\ 0 & \text{if} \quad x \ge \frac{z}{\beta}\,. \end{cases} \tag{2.9}$$

Equation (2.4) straightforwardly transforms into

$$\mu(x,z) = \begin{cases} \lambda \exp(-\nu z) \exp(\nu \beta x) & \text{if} \quad 0 \le x \le \frac{z}{\beta}\,, \\ \lambda & \text{if} \quad x \ge \frac{z}{\beta}\,. \end{cases} \tag{2.10}$$

Environmental parameters

There are arguments for leaving the parameter ν (shock magnitude) time-independent. Thinking of radioactive decay, the number of decaying atoms decreases over time, but the energy magnitude of an emitted particle stays the same. So keeping radioactive half life in mind, for the parameter λ (shock frequency), an expression like $\lambda(y) = \lambda\,\exp(-\varrho y)$ looks sensible. In summary,

$$\mu(x,y,z) = \begin{cases} \lambda\,\exp(\nu\beta x - \varrho y - \nu z) & \text{if} \quad 0 \le x \le \frac{z}{\beta}\,, \\ \\ \lambda\,\exp(-\varrho y) & \text{if} \quad x \ge \frac{z}{\beta}\,. \end{cases} \tag{2.11}$$

Observed force of mortality

What we are finally interested in is $\bar{\mu}(x,y)$, the observed force of mortality of the population at age x and at time y, which is the weighted average of the individual forces of mortality at age x and at time y,

$$\bar{\mu}(x,y) \;=\; \int_0^\infty p(x,y,z)\mu(x,y,z)\,dz\,, \tag{2.12}$$

where $p(x,y,z)$ is the population composition at age x and at time y.

Result

When assuming a relatively concentrated initial distribution of the heterogeneity parameter z, in our case a uniform distribution on $[v_0, V_0]$, the behaviour of $\bar{\mu}(x,y)$ can be obtained from the following approximation.

$$\bar{\mu}(x,y) \approx \begin{cases} \lambda\,\exp(\nu\beta x - \varrho y)\,\frac{\exp(-\nu v_0)-\exp(-\nu V_0)}{\nu(V_0-v_0)} & \text{if} \quad 0 \le x \le \frac{v_0}{\beta}\,, \\ \\ \lambda\,\exp(-\varrho y)\,\frac{\nu(\beta x - v_0)+1-\exp[\nu(\beta x - V_0)]}{\nu(V_0-v_0)} & \text{if} \quad \frac{v_0}{\beta} \le x \le \frac{V_0}{\beta}\,, \\ \\ \lambda\,\exp(-\varrho y) & \text{if} \quad x \ge \frac{V_0}{\beta}\,. \end{cases} \tag{2.13}$$

For fixed y, the right-hand side of (2.13) is a continuous function of x, which is increasing and convex on $\left[0, \frac{v_0}{\beta}\right]$, increasing and concave on $\left[\frac{v_0}{\beta}, \frac{V_0}{\beta}\right]$ and constant on $\left[\frac{V_0}{\beta}, \infty\right)$.

2.3. Illustration

Special case "$v_0 \to V_0$"

As v_0 tends to V_0, the approximation in (2.13) turns into an equation. Since all heterogeneity is lost in the limit, $\bar{\mu}(x, y)$ then equals $\mu(x, y, V_0)$, the force of mortality of an arbitrary individual, given in (2.11). Indeed,

$$\begin{aligned}
\lim_{v_0 \to V_0} \frac{\exp(-\nu v_0) - \exp(-\nu V_0)}{\nu(V_0 - v_0)} &= -\frac{1}{\nu} \lim_{v_0 \to V_0} \frac{\exp(-\nu v_0) - \exp(-\nu V_0)}{v_0 - V_0} \\
&= -\frac{1}{\nu} \left(\frac{d}{dx} \exp(-\nu x) \bigg|_{x = V_0} \right) \\
&= \exp(-\nu V_0) \,, \qquad (2.14)
\end{aligned}$$

so that, by (2.13),

$$\begin{aligned}
\mu(x, y, V_0) &= \bar{\mu}(x, y) \\
&= \begin{cases} \lambda \exp(\nu \beta x - \varrho y) \exp(-\nu V_0) & \text{if} \quad 0 \le x \le \frac{V_0}{\beta} \,, \\ \lambda \exp(-\varrho y) & \text{if} \quad x \ge \frac{V_0}{\beta} \,, \end{cases} \qquad (2.15)
\end{aligned}$$

which is (2.11) with $z = V_0$.

Real data illustration

The expression for the observed mortality rate of a heterogeneous population is given in (2.13). This mortality rate grows according to Gompertz's Law at midlife-ages, then its growth declines, levelling off at high ages. This behaviour is qualitatively consistent with real mortality rates, which is illustrated for period data of Japanese females in the years 1947, 1967, 1987 and 2007.

In the following the expression (2.13) is adapted to real data by choosing the parameters of the model. Namely, the environmental parameters are chosen as $\lambda = 14.4$, $\varrho = 0.0014$ and $\nu = 1$, while the individual parameters are chosen as $\beta = 0.102$, $V_0 = 12.24$ and $v_0 = 8.874$ (in 1947), $v_0 = 9.384$ (in 1967), $v_0 = 9.894$ (in 1987) and $v_0 = 10.404$ (in 2007). In the figures below the black curves correspond to the expression (2.13), while the coloured curves correspond to the real data of different years.

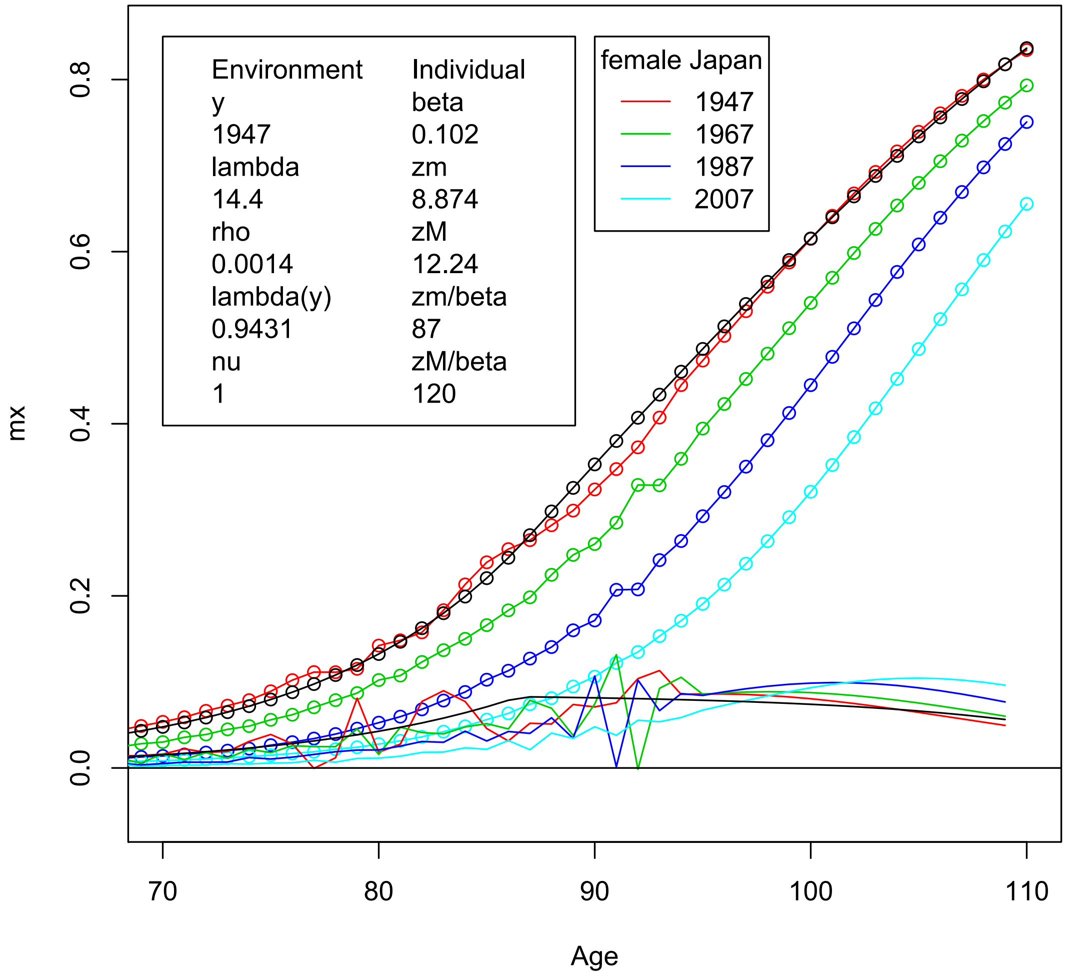

Figure 1: Approximating data for Japanese females in 1947 (black curve). (label gen104)

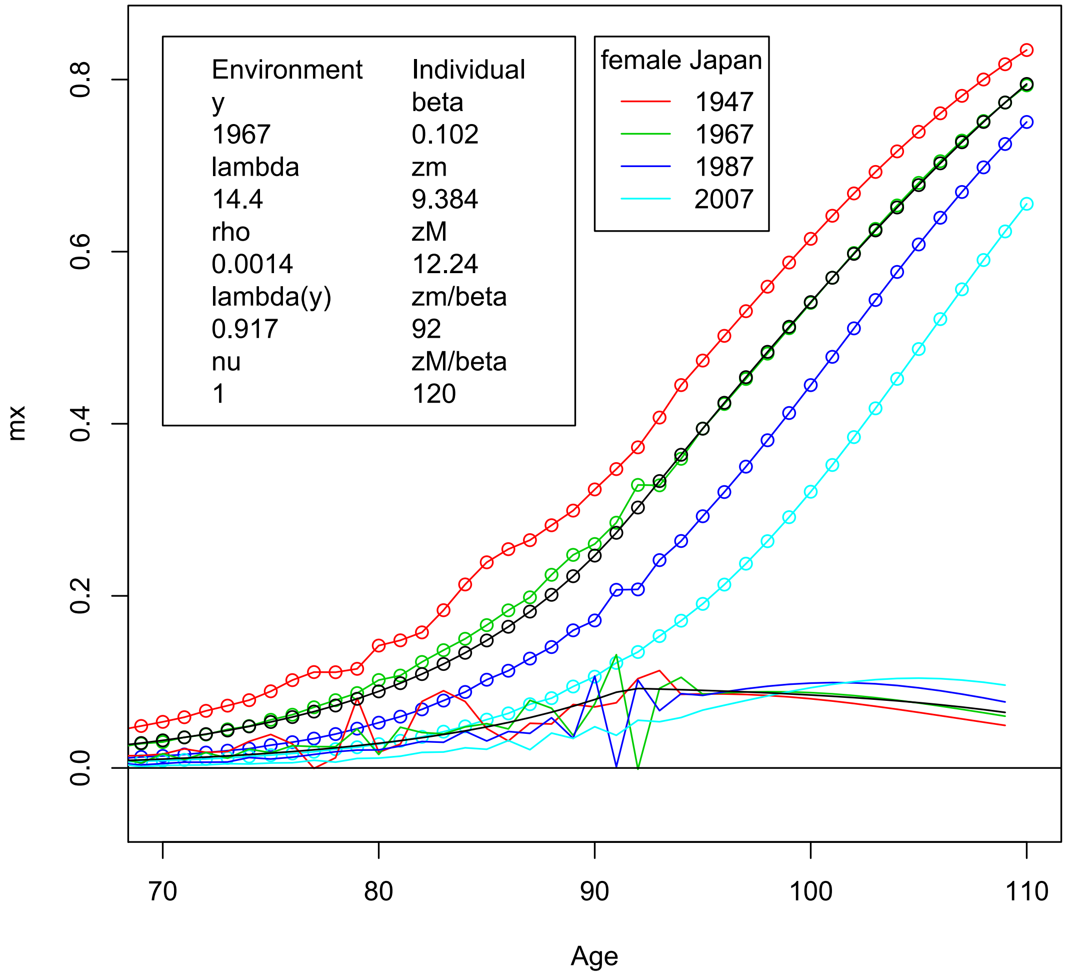

Figure 2: Approximating data for Japanese females in 1967 (black curve). (label gen105)

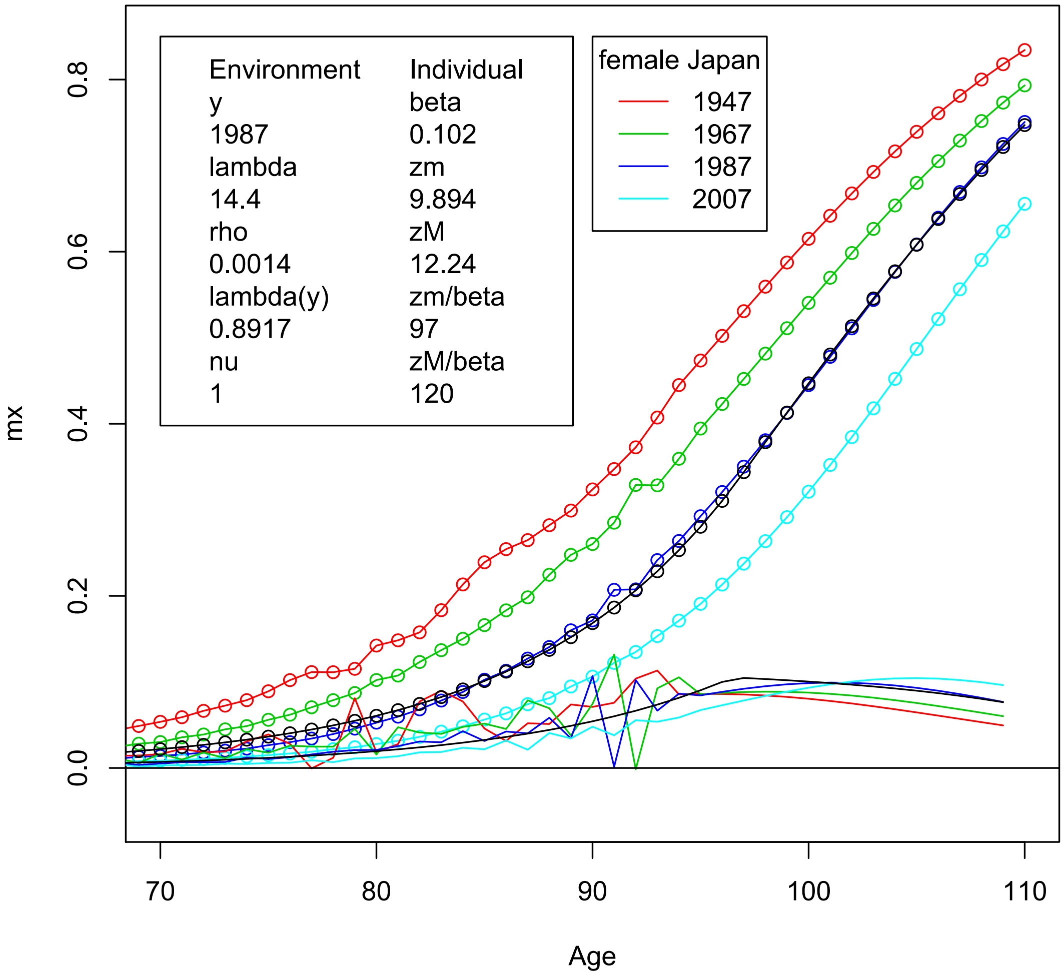

Figure 3: Approximating data for Japanese females in 1987 (black curve). (label gen106)

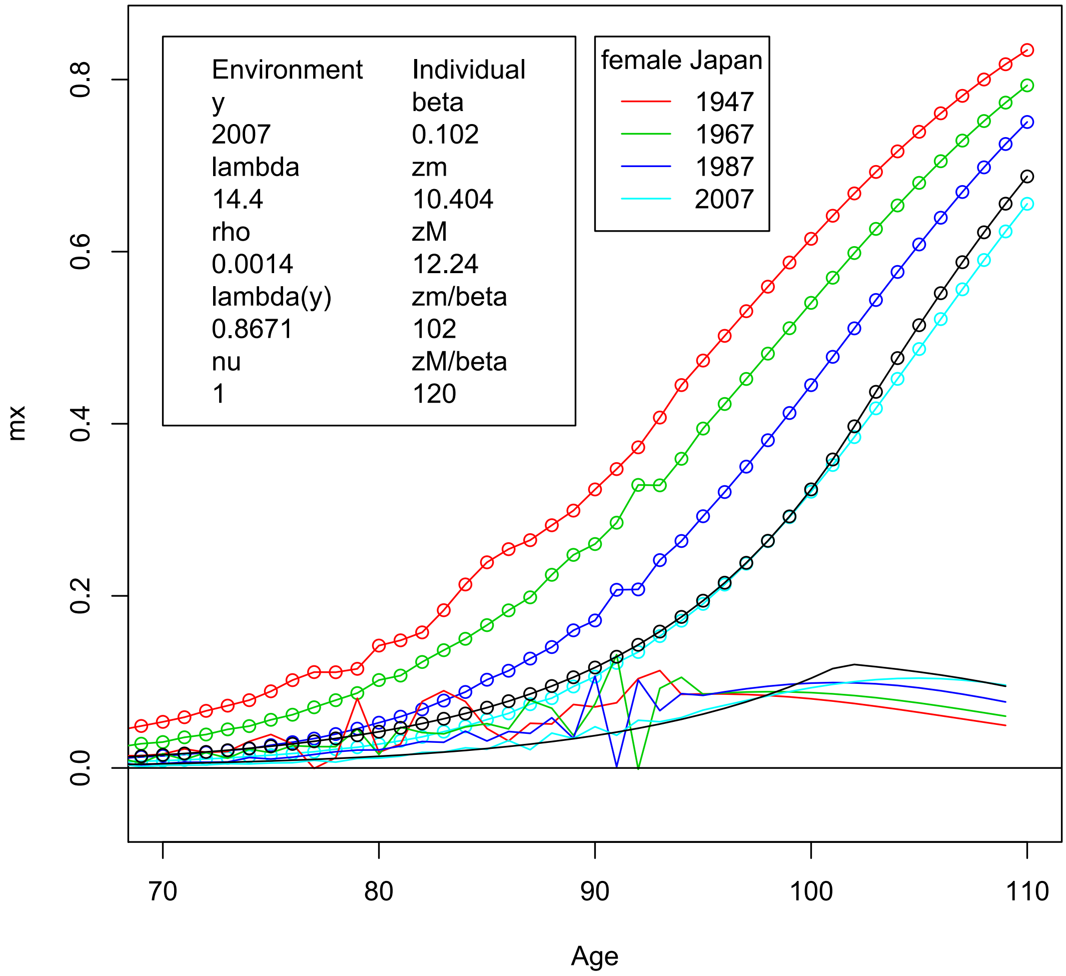

Figure 4: Approximating data for Japanese females in 2007 (black curve). (label gen107)

3. Discussion

3.1. Relationship with Gompertz

Equation (2.2) of Corollary 1, evidently implies the formal equivalence of an exponential force of mortality and a linear vitality. In other words, given Postulates (1) and (2), Gompertz's Law of Mortality and a linear decline of vitality appear to imply each other. This, however, is only valid to a certain extent, since the vitality cannot be negative.

In their original paper [SM60], Strehler and Mildvan essentially postulate equation (2.1) and two possibilities for $\mathbb{P}[M \geq V(x)]$, thereby obtaining equation (2.2) of Corollary 1. They further assume Gompertz-mortality hazards as a part of their theory, while the linear loss of vitality is a prediction thereof. This, however, cannot be done in general, because, by Theorem 1, the force of mortality is bounded above by λ, the parameter characterising the frequency of shocks. Consequently, equation (2.2) of Corollary 1 is simply invalid for negative vitalities, which correspond to forces of mortality exceeding λ.

We prefer to mathematically postulate the concepts of challenges and vitality in Postulate (1) and to formally deduce the expression for the force of mortality in equation (2.1). Postulate (2), concerning the shock magnitudes, is essentially in line with the original second postulate. In contrast, Postulate (3) assumes a linear decline in vitality down to zero and a constant zero vitality from thereon, which, together with Corollary 1, implies a Gompertz-like force of mortality with the key feature of being eventually constant (Corollary 2). Furthermore, the so-called SM correlation can immediately be seen in Corollary 2 and in equations (2.10) and (2.11), with a linearly increasing ν, the parameter characterising the magnitude of shocks, causing a linear decrease of $\log(a)$ and a linear increase of b. Due to the nature of their approach, this special role of ν was not explicitly highlighted in [SM60].

It is interesting to note largely differing interpretations of the original paper in the literature. A concise summary is given in [YIB00]. In [ZYL11], it is suggested that even both Gompertz's Law of Mortality and a linear decline of vitality are part of the theory. On the other hand, it is claimed in [ST62] that the Gompertz-equation is a consequence of the theory. Similarly, in [Atl68], the exponential Gompertz-function is a consequence of the linear decrease of vitality. Finally, this is also suggested in [Fin07]. Finkelstein essentially deduces equation (2.1) of Theorem 1, using theory on point processes. While the exponential form of the mortality rate is then a result of Postulate (2) and linearity in vitality, Postulate (2) is regarded as an "unjustified assumption".

3.2. Vitality concept and decline

Strehler and Mildvan [SM60] introduce the concept of vitality as follows. An organism consists of several subsystems. Each subsystem has some maximum ability to return to initial conditions after a challenge ("a change in condition due to internal or external energy fluctuations"). "Death occurs when the rate at which an organism does work to restore the original state is less than that demanded to overcome the effects of a given challenge." Vitality is "the capacity of an individual organism to stay alive, as measured by an appropriately weighted average of the maximum rate of work output (power output) less the basal power output of all of the functional modalities contributing to survival in the normal environment".

The theoretical concept of vitality can be regarded as a cumulative measure of all physiological functions of a human being. Already then (see, for example, [Sho57]) there was evidence for approximately linear decline of such functions (between 0.5 and 1.3 percent per year) after age 30. See Figure 2 in [SM60], exhibiting an apparent linear decline observed in the eight physiological functions "nerve conduction velocity", "basal metabolic rate", "maximal breathing capacity", "standard cell water", "standard renal plasma flow (Diodrast)", "vital capacity", "standard glomerular filtration rate (inulin)" and "cardiac index". These findings clearly support the assumption of a linearly declining vitality, within a certain period.

The preceding discussion of vitality and its decline inevitably brings up the question of what happens at ages of zero vitality. While this is discussed at the end, we now make a few comments on its decline (and other behaviour).

Our formulation of Postulate (3) is largely driven by its simplicity, expressing the vitality function as consisting of two pieces, a linear function and a constant (zero) function. Incidentally, assuming the constant piece to be positive, by Corollary 1, does not conceptually change the form of the resulting force of mortality. There would still be an initial Gompertz-piece and a constant piece, with the constant being smaller than λ.

As simple as the vitality may be expressed in Postulate (3), its kink at the age of zero vitality is equivalent to a sharp transition of the individual's mortality curve from exponential increase to constancy. While this may or may not be realistic, observed population mortality curves exhibit, at older ages, an initial exponential increase and a gradual levelling off to a (hypothetical) plateau. We obtain this behaviour by introducing heterogeneity. Alternatively, even in a homogeneous population, it would straightforwardly be possible to obtain this behaviour by simply making some helpful assumptions on the vitality, such as postulating a relative decline in vitality, modelling vitality as an exponentially decreasing function.

Indeed, given Corollary 1, any form of the force of mortality, as long as it does not exceed λ, can be obtained by some vitality function, and vice versa. In particular, the force of mortality decreases whenever the vitality increases, as also remarked in [Fin07]. It is further remarked in [Fin07] that the mortality rate decreases whenever λ decreases, which is immediate from Corollary 1.

In [BLZ11], the "standard" SM-model with $\lambda = 1$ and $\nu = 1$ is studied with vitality declining as a power function. It is further assumed that the force of mortality has an age-dependent factor $\Lambda(x)$, which is increasing as a power function, and a Makeham-constant. The model is fitted to mortality data from the 1892 cohort of Swedish women and on more than 1.2 million Mediterranean fruit flies.

3.3. Age of zero vitality

The sole cause of death in the Strehler-Mildvan model is a shock whose magnitude exceeds the current vitality of an organism. The concept of vitality [SM60] discussed above expresses vitality as the difference of the total and the basal power output of an organism. So the total power output is the sum of the basal power output, needed to survive without taking any shocks into account, and the vitality, regarded as an ability to restore initial conditions after a challenge. In particular, vitality should always be non-negative, with its vanishing being irrelevant for survival in the absence of shocks. Vitality being "defined as the capacity of an individual organism to stay alive" can thus be misleading.

Suppose that an organism has zero vitality on an interval of length ε. Then, according to Theorem 1, its force of mortality will be constantly λ on that interval. This surely does not mean immediate death. Certain death, however, is suggested by Strehler and Mildvan's claim [SM60] "that the theory predicts that the maximum lifetime attainable in a homogeneous population will be approximately $1/B$", where "B is the fractional loss each year of original vitality". Also, this claim is omnipresent in [ZYL11]. $1/B$ is the "age of zero vitality", in our Postulate (3) the age from which vitality stays zero. With the force of mortality being constantly λ from age $1/B$, it is straightforward that the remaining life expectancy at age $1/B$ equals $1/\lambda$, and there is, of course, still no guarantee for certain death at or before age $1/B + 1/\lambda$.

The parameter λ, representing the inflow of challenges, is called "the total number of challenges per unit time" in [SM60]. If randomness would be removed from the inflow, so that, for example, it is assumed that there is always a time gap of $1/\lambda$ between two consecutive shocks, then obviously there would be a shock in any interval of length $1/\lambda$. Death for an organism of zero vitality in an interval of length $1/\lambda$ would be certain. However, the force of mortality would conceptually not make much sense any more, with it almost always being zero (whenever it is clear that there are no shocks).

It could be argued that having a "zero vitality" sounds fatal of its own accord, so that death "certainly" has to follow at once. Vitality, however, is just a theoretical quantity which can equal zero without immediate death. In fact, poetically speaking, we imagine vitality as a **shield** giving an organism some partial protection against the strikes of the Grim Reaper. Once the shield has faded away, there is no more protection, but the absence of the shield does not cause immediate death - it is the Reaper's next strike. Once the shield in the fight for life is gone, life continues defencelessly until the next shock.

3.4. Relationship with $\varrho - \gamma -$Gompertz

The ϱ-γ-Gompertz ($\varrho\gamma$G) model has the following four parameters:

- ϱ, the rate at which the force of mortality for individuals is declining over time,
- γ, the variance of the gamma distribution with unit mean for the (individual and proportional) frailty z,
- a, the force of mortality of a standard individual (of unit frailty) at birth and at time zero, and
- b, the rate at which the force of mortality for individuals increases with age (the so-called rate of aging).

An individual of frailty z has the following force of mortality at age x and at time y:

$$\mu(x,y,z) = a\,\exp(bx)\,\exp(-\varrho y)\,z\,.$$

For the population mortality rate, one obtains

$$\bar{\mu}(x,y) \quad = \quad \frac{a\,\exp(bx)\,\exp(-\varrho y)}{1+\gamma\,\frac{a}{b-\varrho}\,[\exp(bx)-\exp(\varrho x)]\,\exp(-\varrho y)}\,.$$

It is straightforward that, for any fixed y, we have

$$\lim_{x\to\infty}\bar{\mu}(x,y) = \frac{b-\varrho}{\gamma}\,.$$

It can be shown that

$$\frac{\partial}{\partial x}\,\bar{\mu}(x,y) = \frac{a\,\exp(bx)\,\exp(-\varrho y)\,\Big[b-\gamma\,a\,\exp(\varrho x)\,\exp(-\varrho y)\Big]}{\Big(1+\gamma\,\frac{a}{b-\varrho}\,[\exp(bx)-\exp(\varrho x)]\,\exp(-\varrho y)\Big)^2}\,.$$

Since

$$b-\gamma\,a\,\exp(\varrho x)\,\exp(-\varrho y) = 0 \iff x = y+\frac{1}{\varrho}\log\left(\frac{b}{\gamma\,a}\right)\,,$$

strictly speaking, for fixed y, $\bar{\mu}(x,y)$ is a strictly increasing function of age x until $x = y+\frac{1}{\varrho}\log\left(\frac{b}{\gamma\,a}\right)$, where it reaches its maximum and from where it is strictly decreasing towards the plateau $\frac{b-\varrho}{\gamma}$. Unfortunately, $\frac{\partial^2}{\partial x^2}\,\bar{\mu}(x,y)$ does not appear to have an expression enabling us to describe the curvature of $\bar{\mu}(x,y)$.

Working with real data, maximum likelihood estimates have been found for the four parameters to fit data of Japanese females from age 80 to 105 and from 1980 ($y = 0$) to 2005. For the obtained values $a = 0.0000006678567$, $b = 0.1456125$, $\varrho = 0.0381227$ and $\gamma = 0.1501849$, $\bar{\mu}(x, 0)$ attains its maximum at $x = 372.17\ldots$, which, of course, cannot possibly be observed in reality.

The fundamental difference between our Strehler-Mildvan (SM) model and the $\varrho\gamma$G-model lies in the form of the individual mortality rates. While our SM-rates are always bounded above by the parameter describing the frequency of the shocks, (cf. (2.11)), the $\varrho\gamma$G-rates grow unboundedly large. Despite the exponential similarity of our SM-rates on an initial age interval and the $\varrho\gamma$G-rates, it is thus impossible to derive a duality between the two models in terms of their parameter sets.

It may still be possible to obtain identical or at least similar population mortalities. However, sensibly introducing heterogeneity into an SM-parameter and making exact calculations is made difficult by the piecewise nature of the SM-rates, which leads to the use of approximations. For example, the approximation in our SM-model displays, on the one hand, the typical three stages of human population mortality curves (cf. (2.13)), but, on the other hand, it does not have a maximum as the $\varrho\gamma$G-curves discussed above.

3.5. Pareto-Distributed Shock Magnitudes

In our SM-model, heterogeneity is introduced via the parameter V_0, describing an individual's vitality at birth. This leads to an individual mortality rate with the heterogeneity variable z occurring in an exponential factor (cf. (2.11)). For obtaining (power) proportionality between an individual's heterogeneity variable and its mortality rate, as it is the case for $\varrho\gamma$G-curves, one could modify Postulate (2) and (3). Let the shock magnitude distribution be Pareto (modification of Postulate (2)):

$$\mathbb{P}(M \geq x) = \begin{cases} \left(\frac{x_m}{x}\right)^{\alpha} & \text{if} \quad x \geq x_m\,, \\ 1 & \text{if} \quad 0 \leq x \leq x_m\,, \end{cases}$$

where $x_m > 0$ and $\alpha > 0$ are some constants. This implies

$$\mu(x) = \begin{cases} \lambda \left(\frac{x_m}{V(x)}\right)^{\alpha} & \text{if} \quad V(x) \geq x_m\,, \\ \lambda & \text{if} \quad 0 \leq V(x) \leq x_m\,. \end{cases} \tag{3.16}$$

Consider the vitality in the form

$$V(x) = V_0 \, \exp(-kx)\,,$$

where $V_0 > 0$ and $k > 0$ are some constants (modification of Postulate (3)). Then (3.16) takes the form

$$\mu(x) = \begin{cases} \lambda \left(\frac{x_m}{V_0 \, \exp(-kx)}\right)^{\alpha} & \text{if} \quad V_0 \, \exp(-kx) \geq x_m\,, \\ \lambda & \text{if} \quad 0 \leq V_0 \, \exp(-kx) \leq x_m\,. \end{cases} \tag{3.17}$$

If

$$x_m \geq V_0 \,,$$

then (3.17) takes the form

$$\mu(x) = \lambda \quad \forall\, x \geq 0\,. \tag{3.18}$$

If

$$x_m < V_0 \,,$$

then (3.17) takes the form

$$\mu(x) = \begin{cases} \lambda \left(\frac{x_m}{V_0}\right)^{\alpha} \exp(\alpha k x) & \text{if} \quad 0 \leq x \leq \frac{1}{k} \log\left(\frac{V_0}{x_m}\right)\,, \\ \\ \lambda & \text{if} \quad x \geq \frac{1}{k} \log\left(\frac{V_0}{x_m}\right)\,. \end{cases} \tag{3.19}$$

Let

$$V_0^{-\alpha} = z\,,$$

where z is Gamma-distributed with mean 1 and variance γ. Then (3.18) and (3.19) imply

$$\mu(x,z) = \begin{cases} z\,\lambda\, x_m^{\alpha} \exp(\alpha k x) & \text{if} \quad z \leq x_m^{-\alpha} \quad \& \quad 0 \leq x \leq \frac{1}{k\alpha} \log\left(\frac{x_m^{-\alpha}}{z}\right)\,, \\ \\ \lambda & \text{if} \quad z \leq x_m^{-\alpha} \quad \& \quad x \geq \frac{1}{k\alpha} \log\left(\frac{x_m^{-\alpha}}{z}\right)\,, \\ \\ \lambda & \text{if} \quad z > x_m^{-\alpha}\,. \end{cases} \tag{3.20}$$

For the special case $\alpha = 1$, it can be seen that, in the first of the three cases in (3.20), an individual's mortality rate is inversely proportional to its vitality at birth. If, for example, the vitality at birth is twice as high for individual A than for individual B, then the force of mortality of individual A is half of the one of individual B. We may thus regard the reciprocal of vitality at birth, similar to the $\varrho\gamma$G-model, as an individual's "frailty" and the heterogeneity variable. While this is a somewhat simpler impact of the heterogeneity variable on the mortality rate than in our SM-model, the conceptual problem with comparing the model with the $\varrho\gamma$G-model, discussed in the previous section, remains the same. The individual mortality rates crucially differ in their boundedness, preventing a duality between the model and the $\varrho\gamma$G-model. Furthermore, despite having the heterogeneity variable as a linear factor, thereby simplifying some necessary integrations, the piecewise nature of the mortality rates given in (3.20) makes exact calculations of population mortalities difficult.

3.6. Exponential Vitality without Heterogeneity

In this section, we discuss a second modification of our SM-model. We study the impact of exponentially declining vitality (modification of Postulate (3)) on the mortality rate in a homogeneous population. The advantage of postulating a relative decline in vitality is mainly the fact that vitality never reaches zero, thereby avoiding a possibly controversial scenario. Furthermore, the vitality initially declining approximately linearly and then approaching zero more and more slowly yields a behaviour of the force of mortality, which is similar to the approximation in our SM-model. In particular, the mortality curve has a convex piece, a point of inflexion, a concave piece and a plateau, without having any kinks, which are so typical for our individual SM-rates. Indeed, suppose that the vitality $V(x)$ at age x satisfies

$$V(x) = V_0 \, \exp(-B'x)\,,$$

where $V_0 > 0$ and $B' > 0$ are some constants. Then, by Corollary 1,

$$\mu(x) \;=\; \lambda \, \exp[-\nu V(x)] \;=\; \lambda \, \exp[-\nu V_0 \, \exp(-B'x)]\,. \tag{3.21}$$

Suppose further that ν, the parameter describing the magnitudes of the shocks, grows relatively over time. In other words, the expected value of the shock magnitude being the reciprocal of ν, we assume that this expectation declines relatively over time:

$$\nu(y) = \nu_0 \, \exp(\varrho' y)\,,$$

where $\nu_0 > 0$ and $\varrho' \geq 0$ are some constants. Then, by (3.21),

$$\begin{aligned} \mu(x,y) &= \lambda \, \exp[-\nu_0 \, \exp(\varrho' y) \, V_0 \, \exp(-B'x)] \\ &= \lambda \, \exp[-\eta(y) \, \exp(-B'x)]\,, \end{aligned} \tag{3.22}$$

where $\eta(y) = \nu_0 \, \exp(\varrho' y) \, V_0$. Then

$$\frac{\partial}{\partial x} \, \mu(x,y) = \mu(x,y) \, \Big[\eta(y) B' \, \exp(-B'x)\Big] > 0\,,$$

and

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \, \mu(x,y) &= \mu(x,y) \, \Big[\eta(y) B' \, \exp(-B'x)\Big] \, \Big[\eta(y) B' \, \exp(-B'x) - B'\Big] \\ &= \mu(x,y) \, \Big[\eta(y) B'^2 \, \exp(-B'x)\Big] \, \Big[\eta(y) \, \exp(-B'x) - 1\Big]\,, \end{aligned}$$

with

$$\eta(y) \, \exp(-B'x) - 1 = 0 \iff x = \frac{\log[\eta(y)]}{B'} \iff x = \frac{\varrho' y + \log(\nu_0 V_0)}{B'}\,.$$

Thus, for fixed time y, $\mu(x,y)$ is a strictly increasing function of x, asymptotically approaching λ. Furthermore, it has a point of inflexion at $\frac{\varrho' y + \log(\nu_0 V_0)}{B'}$, before which it is strictly convex and after which it is strictly concave.

Comparing this model with the $\varrho\gamma$G-model, we should set $\lambda = (b - \varrho)/\gamma$ for getting equal plateaus. Assuming $a = 0.0000006678567$, $b = 0.1456125$, $\varrho = 0.0381227$, $\gamma = 0.1501849$ and $y = 0$ (in the year 1980 for Japanese females), it appears that $\lambda = 0.7157\ldots$, $\nu_0 V_0 = 5,300$, $\varrho' = 0.025$ and $B' = 0.095$ provide a good approximation of the $\varrho\gamma$G-mortality surface between the years 1980 and 2005 and the ages 80 and 105. The point of inflexion in 1980 for those parameter values, for example, would approximately be at age 90.3.

The relative growth of the mortality rates between ages 30 and 80, however, is not constant (corresponding to an exponential absolute growth), but strictly decreasing. One might argue, though, that we should primarily use this model for "senescent" mortality, almost exclusively applying for age 80 and above, and that "middle" mortality plays a much bigger role for the exponential growth between ages 30 and 80. Therefore, it would still be interesting to find maximum likelihood estimates for the parameters of this model (cf. (3.22)), say for data of Japanese females from age 80 to 105 and from 1980 ($y = 0$) to 2005.

3.7. Discrete to Continuous – Arriving at Gompertz

In the "standard" SM-model (and also in the "Pareto-modification" discussed above), it is possible to restrict the range of the vitality parameters in such a way that an exponential mortality rate is obtained for all individuals on an initial age interval. However, when setting $\lambda = 0.7$, in an attempt to resemble the plateau, and choosing the initial age interval to be up to age 120, any obtained exponential function is far too small (or flat) at middle ages to represent real data. Furthermore, as for heterogeneity, the restriction would affect the finding of an interesting distribution for z (vitality at birth) and the necessary calculations for obtaining a population force of mortality.

When setting λ to be larger, say $\lambda = 7$, it would not represent the plateau any more but an absolute upper bound for an individual, which he or she might reach earlier or later in life, depending on the vitality at birth. At least, some of the obtained exponential mortality rates would have a reasonable trajectory between ages 30 and 80, while the eventual levelling off at around 0.7 would have to be achieved, theoretically, by heterogeneity calculations.

By Theorem 1, the force of mortality in the SM-theory is always bounded above by λ, the parameter characterising the frequency of shocks. For possibly obtaining an unbounded Gompertz-curve on an invidivual basis, λ would theoretically have to be sent to infinity, thereby crossing the threshold from a **discrete** occurrence of shocks **to** a **continuous** omnipresent shock-level. Such an attempt would be at the cost of Postulate (1), the main building block of the "standard" SM-theory, thereby losing Theorem 1 and retaining the mere concept of survival being governed by the rivalry "environmental energy against individual vitality".

Indeed, consider a small age interval $[x, x + \varepsilon]$ of width ε starting at age x.

Suppose that an individual experiences an amount

$$E = \int_x^{x+\varepsilon} \eta(y)\,dy$$

of destructive energy, where η denotes its density. Suppose that an individual's vitality V is approximately constant on the interval. The occurrence of death is not decided by the simple comparison of a shock magnitude and vitality, as in the "standard" SM-theory. Instead, the interplay of energy and vitality needs to be redefined. We assume that an individual of vitality V who experiences destructive energy E has survival probability $p(E, V)$. Now two crucial assumptions about $p(E, V)$ are made. First, the survival probability of an individual of vitality V, experiencing a total amount of $E_1 + E_2$ of destructive energy, should be the product of the two survival probabilities given by experiencing E_1 and E_2 separately; that is,

$$p(E_1 + E_2, V) = p(E_1, V)\,p(E_2, V)\,. \tag{3.23}$$

Second, the survival probability of an individual of vitality V, experiencing an amount of E of destructive energy, should, for any positive real number α, equal the survival probability of an individual of vitality αV, experiencing an amount of αE of destructive energy; that is,

$$p(\alpha E, \alpha V) = p(E, V)\,. \tag{3.24}$$

We can now calculate $\mu(x)$, the force of mortality at age x. It follows from the theory of functions (Cauchy's functional equation), that, for any fixed V, the only continuous solution of equation (3.23) is

$$p(E, V) = p(1, V)^E\,. \tag{3.25}$$

By equations (3.24) and (3.25), we have

$$p(1, V) = p(1/V, 1) = p(1, 1)^{1/V}\,,$$

so that

$$p(E, V) = p(1, 1)^{E/V} = \exp\{\,\log[p(1, 1)]\,E/V\,\}\,. \tag{3.26}$$

By definition,

$$\mu(x) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}[1 - p(E, V)]\bigg|_{\varepsilon=0}\,. \tag{3.27}$$

Suppose, for simplicity, that η is constant, so that $E = \eta\varepsilon$. Then it follows from equations (3.27) and (3.26) that

$$\begin{aligned} \mu(x) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}[1 - \exp\{\,\log[p(1, 1)]\,\eta\varepsilon/V\,\}]\bigg|_{\varepsilon=0} \\ &= -\log[p(1, 1)]\,\eta/V \\ &= \log[1/p(1, 1)]\,\eta/V\,. \end{aligned} \tag{3.28}$$

For convenience, we may assume that vitality is such that "one unit of vitality against one unit of destructive energy" results in $p(1,1) = \exp(-1)$, getting rid of $\log[1/p(1,1)]$ in equation (3.28); that is,

$$\mu(x) = \eta/V\,.$$

So, if $V(x) = V_0 \exp(-\beta x)$, we obtain the individual Gompertz-mortality

$$\mu(x) = \frac{\eta}{V_0}\exp(\beta x)\,.$$

For a **duality** with the ϱ-γ-Gompertz model, ϱ could be introduced via the environmental parameter η, whereas a Gamma-distribution with mean 1 and variance γ could be assumed for the reciprocal of the individual parameter V_0 (vitality at birth), since "vitality" and "frailty" are inversely proportional. It is thus possible to establish a duality between a continuous SM-version and the $\varrho-\gamma-$Gompertz model.

4. Proofs and derivations

4.1. "Standard" SMT

Proof of Theorem 1

Let us select an arbitrary individual and consider its force of mortality. In this proof, for simplicity, we assume that the notions of age and time coincide, that is, that the individual was born at time zero. We use the variable t for both age and time. Let X be the random variable for the age of death of the individual. The force of mortality $\mu(t)$ at age t is defined as the limit of the ratio of the conditional probability of death in the age interval $[t, t+\varepsilon]$ given survival until age t and the width $\varepsilon > 0$ of the interval, as ε tends to zero,

$$\begin{aligned}\mu(t) &= \lim_{\varepsilon\to 0}\left(\frac{\mathbb{P}(\,t \le X \le t+\varepsilon \,|\, X \ge t\,)}{\varepsilon}\right)\\ &= \lim_{\varepsilon\to 0}\left(\frac{1-\mathbb{P}(\,X \ge t+\varepsilon \,|\, X \ge t\,)}{\varepsilon}\right)\,. \qquad (4.29)\end{aligned}$$

Let us have a closer look at $\mathbb{P}(\,X \ge t+\varepsilon \,|\, X \ge t\,)$. Given survival until age t, the conditional probability of survival in the age interval $[t, t+\varepsilon]$ is determined by the shocks in that interval. Remember that the environmental shocks arrive universally, in a sense that at any given point in time, there has always been a shock before and will always be a shock after. Let t_0 be the time of the last shock before time t. Let t_i, for $i = 1, 2, ...$, be the time of the i^{th} shock after (or at) time t. Let $\tau_1' = t_1 - t_0$ and, for $i = 2, 3, ...$, $\tau_i = t_i - t_{i-1}$. We know from Postulate (1) that τ_1' and τ_i, for $i = 2, 3, ...$, are independent and exponentially distributed random variables with parameter λ.

Let $\tau_1 = t_1 - t$. Let f_{t_0} be the probability density function of t_0. Then, by the memoryless property of the exponential distribution, we have

$$\begin{aligned}
\mathbb{P}(\tau_1 \geq s) &= \mathbb{P}(t + \tau_1 \geq t + s) \;=\; \mathbb{P}(t_0 + \tau_1' \geq t + s \,|\, t_0 + \tau_1' \geq t) \\
&= \int_{-\infty}^{t} \mathbb{P}(t_0 + \tau_1' \geq t + s \,|\, t_0 + \tau_1' \geq t,\, t_0 = u) f_{t_0}(u)\, du \\
&= \int_{-\infty}^{t} \mathbb{P}(u + \tau_1' \geq t + s \,|\, u + \tau_1' \geq t) f_{t_0}(u)\, du \\
&= \int_{-\infty}^{t} \mathbb{P}(\tau_1' \geq t - u + s \,|\, \tau_1' \geq t - u) f_{t_0}(u)\, du \\
&= \int_{-\infty}^{t} \mathbb{P}(\tau_1' \geq s) f_{t_0}(u)\, du \;=\; \mathbb{P}(\tau_1' \geq s) \qquad (4.30)
\end{aligned}$$

Thus τ_1 is also exponentially distributed with parameter λ, and, for $j \geq 1$, we have $t_i = t + \sum_{j=1}^{i} \tau_j$.

Let $N(t, \varepsilon)$ denote the total number of challenges in the interval $[t, t + \varepsilon]$. By the Law of Total Probability, we have

$$\begin{aligned}
&\mathbb{P}(\, X \geq t + \varepsilon \,|\, X \geq t \,) = \\
&\quad \sum_{i=0}^{\infty} \mathbb{P}(\, X \geq t + \varepsilon \,|\, X \geq t \,,\, N(t,\varepsilon) = i \,)\, \mathbb{P}(\, N(t,\varepsilon) = i \,) \\
&= \sum_{i=0}^{\infty} \left(\prod_{j=1}^{i} \mathbb{P}(M \leq V(t_j)) \right) \mathbb{P}(\, N(t,\varepsilon) = i \,)\,. \qquad (4.31)
\end{aligned}$$

A simple property from the theory of Poisson processes is stated in the following lemma.

Lemma 1 *For all $i \geq 0$, we have*

$$\mathbb{P}(\, N(t,\varepsilon) = i \,) = \exp(-\lambda\varepsilon) \frac{(\lambda\varepsilon)^i}{i!}\,. \qquad (4.32)$$

By Postulate (1), V is a continuous function and thus attains its minimum value $V_{\min}(t,\varepsilon)$ and maximum value $V_{\max}(t,\varepsilon)$ on $[t, t+\varepsilon]$. Furthermore, by continuity, we have

$$\lim_{\varepsilon \to 0} V_{\min}(t,\varepsilon) = V(t) = \lim_{\varepsilon \to 0} V_{\max}(t,\varepsilon)\,. \qquad (4.33)$$

By equation (4.31) and Lemma 1, it follows that

$$\begin{aligned}
\mathbb{P}(\, X \geq t + \varepsilon \,|\, X \geq t \,) &\geq \sum_{i=0}^{\infty} \mathbb{P}(M \leq V_{\min}(t,\varepsilon))^i \exp(-\lambda\varepsilon) \frac{(\lambda\varepsilon)^i}{i!} \\
&= \exp(-\lambda\varepsilon) \exp(\lambda\varepsilon \mathbb{P}(M \leq V_{\min}(t,\varepsilon))) \\
&= \exp(\lambda\varepsilon(\mathbb{P}(M \leq V_{\min}(t,\varepsilon)) - 1)) \\
&= \exp(-\lambda\varepsilon \mathbb{P}(M \geq V_{\min}(t,\varepsilon))) \qquad (4.34)
\end{aligned}$$

and, analogously,

$$\begin{aligned}\mathbb{P}(\,X \geq t+\varepsilon \,|\, X \geq t\,) &\leq \sum_{i=0}^{\infty} \mathbb{P}(M \leq V_{\max}(t,\varepsilon))^i \exp(-\lambda\varepsilon)\frac{(\lambda\varepsilon)^i}{i!} \\ &= \exp(-\lambda\varepsilon\mathbb{P}(M \geq V_{\max}(t,\varepsilon)))\,. \qquad (4.35)\end{aligned}$$

By Postulate (1), M has a continuous distribution function. Hence, by equations (4.29), (4.33), (4.34) and (4.35),

$$\begin{aligned}\lambda\mathbb{P}(M \geq V(t)) &= \lim_{\varepsilon\to 0}\left(\frac{1-\exp(-\lambda\varepsilon\mathbb{P}(M \geq V_{\max}(t,\varepsilon)))}{\varepsilon}\right) \\ &\leq \lim_{\varepsilon\to 0}\left(\frac{1-\mathbb{P}(\,X \geq t+\varepsilon \,|\, X \geq t\,)}{\varepsilon}\right) = \mu(t) \\ &\leq \lim_{\varepsilon\to 0}\left(\frac{1-\exp(-\lambda\varepsilon\mathbb{P}(M \geq V_{\min}(t,\varepsilon)))}{\varepsilon}\right) \\ &= \lambda\mathbb{P}(M \geq V(t))\,, \qquad (4.36)\end{aligned}$$

which finishes the proof.

4.2. Heterogeneity (and time-dependence)

By equations (2.12) and (2.11), for calculating $\bar{\mu}(x,y)$, we need to find $p(x,y,z)$, the z-composition of the population of age x at time y. Under the assumption that the z-composition $p(0,y-x,z)$ of the population of the newborns at time $y-x$ is independent of $y-x$, we have

$$p(x,y,z) = \frac{p(0,y-x,z)l(x,y,z)}{\int_0^\infty p(0,y-x,z)l(x,y,z)\,dz} = \frac{p(0,0,z)l(x,y,z)}{\int_0^\infty p(0,0,z)l(x,y,z)\,dz}\,, \qquad (4.37)$$

where $l(x,y,z)$ is the probability that an individual with vitality intercept z survives until age x at time y. In other words, $l(x,y,z)$ is the probability that an individual born at time $y-x$ with vitality intercept z survives until age x, and we have

$$l(x,y,z) = \exp\left(-\int_0^x \mu(t,y-x+t,z)\,dt\right)\,. \qquad (4.38)$$

Using equations (4.38) and (2.11), a fairly complicated expression can be calculated for $l(x,y,z)$. However, for obtaining $p(x,y,z)$, it appears that the necessary integrations cannot be performed exactly any more, whatever continuous distribution $p(0,0,z)$ one might try in equation (4.37). Still, we may try to catch the behaviour of $\bar{\mu}(x,y)$ by assuming a relatively concentrated initial distribution $p(0,0,z)$, resulting in fairly equal $l(x,y,z)$ and the $p(x,y,z)$ being about $p(0,0,z)$. Then, by equation (2.12),

$$\begin{aligned}\bar{\mu}(x,y) &= \int_0^\infty p(x,y,z)\mu(x,y,z)\,dz \\ &\approx \int_0^\infty p(0,0,z)\mu(x,y,z)\,dz\,. \qquad (4.39)\end{aligned}$$

Indeed, let us suppose that z is distributed uniformly on $[v_0, V_0]$ at birth, that is,

$$p(0,0,z) = \begin{cases} 0 & \text{if} \quad 0 \le z < v_0\,, \\[2ex] \frac{1}{V_0 - v_0} & \text{if} \quad v_0 \le z \le V_0\,, \\[2ex] 0 & \text{if} \quad z > V_0\,. \end{cases} \tag{4.40}$$

Then

$$\bar{\mu}(x,y) \approx \int_0^\infty p(0,0,z)\mu(x,y,z)\,dz = \frac{1}{V_0 - v_0}\int_{v_0}^{V_0} \mu(x,y,z)\,dz\,. \tag{4.41}$$

Three cases need to be considered. If $0 \le x \le \frac{v_0}{\beta}$, by (2.11), we have

$$\begin{aligned} \bar{\mu}(x,y) &\approx \frac{1}{V_0 - v_0}\int_{v_0}^{V_0} \mu(x,y,z)\,dz \\ &= \frac{1}{V_0 - v_0}\int_{v_0}^{V_0} \lambda\,\exp(\nu\beta x - \varrho y - \nu z)\,dz \\ &= \frac{\lambda\,\exp(\nu\beta x - \varrho y)}{V_0 - v_0}\int_{v_0}^{V_0} \exp(-\nu z)\,dz \\ &= \lambda\,\exp(\nu\beta x - \varrho y)\,\frac{\exp(-\nu v_0) - \exp(-\nu V_0)}{\nu(V_0 - v_0)}\,. \end{aligned} \tag{4.42}$$

If $\frac{v_0}{\beta} \le x \le \frac{V_0}{\beta}$, we have

$$\begin{aligned} \bar{\mu}(x,y) &\approx \frac{1}{V_0 - v_0}\int_{v_0}^{V_0} \mu(x,y,z)\,dz \\ &= \frac{1}{V_0 - v_0}\left(\int_{v_0}^{\beta x} \mu(x,y,z)\,dz + \int_{\beta x}^{V_0} \mu(x,y,z)\,dz\right) \\ &= \frac{1}{V_0 - v_0}\left(\int_{v_0}^{\beta x} \lambda\,\exp(-\varrho y)\,dz + \int_{\beta x}^{V_0} \lambda\,\exp(\nu\beta x - \varrho y - \nu z)\,dz\right) \\ &= \frac{\lambda\,\exp(-\varrho y)}{V_0 - v_0}\left(\int_{v_0}^{\beta x} 1\,dz + \int_{\beta x}^{V_0} \exp[\nu(\beta x - z)]\,dz\right) \\ &= \frac{\lambda\,\exp(-\varrho y)}{V_0 - v_0}\left((\beta x - v_0) + \frac{1}{\nu}(1 - \exp[\nu(\beta x - V_0)])\right) \\ &= \lambda\,\exp(-\varrho y)\,\frac{\nu(\beta x - v_0) + 1 - \exp[\nu(\beta x - V_0)]}{\nu(V_0 - v_0)}\,. \end{aligned} \tag{4.43}$$

Finally, if $x \ge \frac{V_0}{\beta}$, we have

$$\begin{aligned} \bar{\mu}(x,y) &\approx \frac{1}{V_0 - v_0}\int_{v_0}^{V_0} \mu(x,y,z)\,dz \\ &= \frac{1}{V_0 - v_0}\int_{v_0}^{V_0} \lambda\,\exp(-\varrho y)\,dz \\ &= \lambda\,\exp(-\varrho y)\,, \end{aligned} \tag{4.44}$$

which completes the derivation of (2.13).

4.3. Proof of the Properties

Let us denote the right-hand side of (2.13), regarded as a function of x only, with $\varphi(x)$. Clearly, φ is continuous on the three intervals $\left[0, \frac{v_0}{\beta}\right)$, $\left(\frac{v_0}{\beta}, \frac{V_0}{\beta}\right)$ and $\left(\frac{V_0}{\beta}, \infty\right)$. Checking the situation at $x = \frac{v_0}{\beta}$,

$$\lambda \exp(\nu v_0 - \varrho y) \frac{\exp(-\nu v_0) - \exp(-\nu V_0)}{\nu(V_0 - v_0)} = \lambda \exp(-\varrho y) \frac{1 - \exp[\nu(v_0 - V_0)]}{\nu(V_0 - v_0)},$$

and at $x = \frac{V_0}{\beta}$,

$$\lambda \exp(-\varrho y) \frac{\nu(V_0 - v_0) + 1 - \exp(0)}{\nu(V_0 - v_0)} = \lambda \exp(-\varrho y),$$

we find that φ is continuous on $[0, \infty)$.

Let us denote the first derivative of φ with φ'. We have

$$\varphi'(x) = \begin{cases} \lambda\beta \exp(\nu\beta x - \varrho y) \frac{\exp(-\nu v_0) - \exp(-\nu V_0)}{V_0 - v_0} & \text{if} \quad 0 \le x \le \frac{v_0}{\beta}, \\ \lambda\beta \exp(-\varrho y) \frac{1 - \exp[\nu(\beta x - V_0)]}{V_0 - v_0} & \text{if} \quad \frac{v_0}{\beta} \le x \le \frac{V_0}{\beta}, \\ 0 & \text{if} \quad x \ge \frac{V_0}{\beta}. \end{cases} \tag{4.45}$$

Clearly, φ' is continuous on the three intervals $\left[0, \frac{v_0}{\beta}\right)$, $\left(\frac{v_0}{\beta}, \frac{V_0}{\beta}\right)$ and $\left(\frac{V_0}{\beta}, \infty\right)$. Checking the situation at $x = \frac{v_0}{\beta}$,

$$\lambda\beta \exp(\nu v_0 - \varrho y) \frac{\exp(-\nu v_0) - \exp(-\nu V_0)}{V_0 - v_0} = \lambda\beta \exp(-\varrho y) \frac{1 - \exp[\nu(v_0 - V_0)]}{V_0 - v_0},$$

and at $x = \frac{V_0}{\beta}$,

$$\lambda\beta \exp(-\varrho y) \frac{1 - \exp(0)}{V_0 - v_0} = 0,$$

we find that also φ' is continuous on $[0, \infty)$.

Finally, by (4.45), φ' is positive and strictly increasing on $\left[0, \frac{v_0}{\beta}\right]$, positive and strictly decreasing on $\left[\frac{v_0}{\beta}, \frac{V_0}{\beta}\right)$ and zero on $\left[\frac{V_0}{\beta}, \infty\right)$. Thus φ is strictly increasing and convex on $\left[0, \frac{v_0}{\beta}\right]$, strictly increasing and concave on $\left[\frac{v_0}{\beta}, \frac{V_0}{\beta}\right]$ and constant on $\left[\frac{V_0}{\beta}, \infty\right)$, so that there is a point of inflection at $x = \frac{v_0}{\beta}$ and a levelling off to a plateau until $x = \frac{V_0}{\beta}$.

5. Summary and Outlook

In this paper the Strehler-Mildvan theory of mortality and aging [SM60] is generalised to cover heterogeneity effects in the population. The theory is based on the concept of environmental shocks that cause death of an individual when exceeding its vitality. Heterogeneity is introduced via the value of the vitality of an individual at birth. The main result of the paper is an expression for the observed mortality rate of the heterogeneous population. This mortality rate grows according to Gompertz's Law up to an inflection point, then its growth declines, reaching a constant plateau at the end. This behaviour is qualitatively consistent with real mortality rates, which is illustrated for period data of Japanese females in the years 1947, 1967, 1987 and 2007. Finally, the duality between a continuous SM-version and the $\varrho-\gamma-$Gompertz model is discussed.

The approximate formula for the observed mortality rate provides quite promising results, when applied to real data. Nevertheless, it would be of interest to study the accuracy of the approximation in more detail, perhaps by comparing it to a more precise calculation obtained by numerical integration. The adjustment of the parameters in the expression of the mortality rate to real data has been done "by hand". An automatic procedure, for example in the spirit of "least squares", would be desirable when dealing with more extensive data sets. It should be, however, taken into account that data for mortality rates at older ages are less reliable and obtained by various smoothing mechanisms. The point of inflection in the observed mortality rates moves with time. In the present context this has been taken into account by adjusting the distribution of the heterogeneity parameter. It would be a theoretical challenge to obtain a moving point of inflection directly from the time-dependence of the model parameters.

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References

[Atl68] Henri Atlan. Strehler's Theory of Mortality and the Second Principle of Thermodynamics. *The Journal of Gerontology*, 23(2):196–200, 1968.

[BLZ11] Mark Bebbington, Chin-Diew Lai, and Ričardas Zitikis. Modelling Deceleration in Senescent Mortality. *Mathematical Population Studies*, 18:18–37, 2011.

[Fin07] Maxim Finkelstein. Aging: Damage accumulation versus increasing mortality rate. *Mathematical Biosciences*, 207:104–112, 2007.

[RCGV05] Jean-Marie Robine, Amandine Cournil, Jutta Gampe, and James W. Vaupel. IDL, the International Database on Longevity. *Presented at The Living to 100 and Beyond Symposium Sponsored by the Society of Actuaries Orlando, Fla. January 12-14, 2005*, 2005.

[Sho57] N. W. Shock. Age Changes in Some Physiologic Processes. *Survey of Anesthesiology*, 1(6):619–620, 1957.

[SM60] Bernard L. Strehler and Albert S. Mildvan. General Theory of Mortality and Aging. *Science*, 132(3418):14–21, 1960.

[ST62] G. A. Sacher and E. Trucco. The Stochastic Theory of Mortality. *Annals of the New York Academy of Sciences*, 96:985–1007, 1962.

[YBB$^+$01] Anatoli I. Yashin, Alexander S. Begun, Serge I. Boiko, Svetlana V. Ukraintseva, and Jim Oeppen. The new trends in survival improvement require a revision of traditional gerontological concepts. *Experimental Gerontology*, 37:157–167, 2001.

[YIB00] Anatoli I. Yashin, Ivan A. Iachine, and Alexander S. Begun. Mortality modeling: A review. *Mathematical Population Studies*, 8(4):305–332, 2000.

[ZYL11] Hui Zheng, Yang Yang, and Kenneth C. Land. Heterogeneity in the Strehler-Mildvan General Theory of Mortality and Aging. *Demography*, 48:267–290, 2011. 10.1007/s13524-011-0013-8.