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Tim Riffe I tim.riffe@ehu.eus
Francisco Villavicencio
Mauricio González-Forero

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Sensitivity and decomposition of multistate healthy life expectancy

Tim Riffe*[1,2,3], Francisco Villavicencio†[4], and Mauricio González-Forero‡[5]

[1]OPIK, Departamento de Sociologia y Trabajo Social, Universidad del País Vasco (UPV/EHU)
[2]Ikerbasque (Basque Foundation for Science)
[3]Laboratory of Population Health, Max Planck Institute for Demographic Research
[4]Department of Economic, Financial and Actuarial Mathematics, University of Barcelona
[5]Unaffiliated

Abstract

Background Previous research has proposed an analytic method to decompose healthy life expectancy (HLE) in discrete-time multistate frameworks, which relies on a particular parameterization for calculation. No published work has considered how different HLE parameterizations might give inconsistent decomposition results and interpretations.

Objective We aim to explain (i) why HLE sensitivity and decomposition results are different between three specific multistate parameterizations, (ii) how to translate decomposition results between different parameterizations, and (iii) to propose the use of one parameterization for the interpretation of HLE decompositions.

Methods We compute the analytic sensitivities for three different HLE parameterizations by applying formulas for the sensitivity of a recurrence. This enables us to decompose HLE using the life table response experiment approach and analytically compare the three parameterizations. For our example data, we derive the transition probabilities between health states from coefficients and formulas available in the literature, which summarize disability and mortality in the USA between 1986 and 1990. With these transitions, we calculate parameter sensitivities and decompose the sex gap in HLE under the three different parameterizations.

Results We obtain disability-free and disabled life expectancies (DFLE, DLE, respectively) and their sensitivities and decompositions under three parameterizations. We show how the choice of parameterization affects the interpretation of decomposition results on sex differences in DFLE (DLE). We give formulas to translate the sensitivity results between parameterizations.

Conclusions Researchers should consider the choice of parameterization when calculating the sensitivity or decomposition of a recurrence. We suggest the use of an attrition-based parameterization when interpreting HLE sensitivity and decomposition.

Introduction

Healthy life expectancy (HLE), a measure of a population's average years in good health, is of primary importance in contemporary public health monitoring and demographic research. HLE is often calculated by combining information from a life table and the prevalence of a health state, the so-called Sullivan method [Sullivan, 1971]. Multistate models of HLE offer a representation of health dynamics based on transitions between health states and mortality risks differentiated by health states. Demographic decomposition is a tool to help understand what accounts for the differences between two populations in summary measures, such as HLE. Decompositions of differences in Sullivan HLE partition differences into prevalence and mortality components [Nusselder and Looman, 2004, Shkolnikov et al., 2017], but they are unable to determine how much of a difference is due to onset versus recovery from a health condition, or how much is due to mortality differentiated by health state. Decomposition of multistate HLE tells us which ages and transitions matter for explaining differences between populations. Such decomposition results can tell us which transitions to alter or improve to narrow such inequalities.

Shen et al. [2023] propose an analytic method to decompose discrete-time multistate indices, such as HLE, into the respective contributions of health transitions. This method is an instance of the Life Table Response Experiment approach to decomposition [Caswell, 1989], which is based on the sensitivity of survivorship to transition parameters. Shen's method is designed for a specific HLE parameterization used in matrix algebra calculations [Caswell and van Daalen, 2021], which does not explicitly rely on transitions to death. It yields a decomposition result and interpretation that are specific to and internally consistent for this mortality-free parameterization.

In any multistate model, different parameterizations of the age-dynamics of health survivorship can be used to obtain the same HLE estimate. At first glance trivial, this observation is consequential when decomposing

*tim.riffe@ehu.eus
†villavicencio@ub.edu
‡Mauricio.GonzalezForero@gmail.com

HLE differences. Reliable and consistent decomposition results are of utmost importance if demographic decomposition is to be used as a means to identify priorities among interventions designed to modify health transitions and mortality. We show that the parameterization chosen to decompose differences in HLE affects the estimated size and sign of the response of HLE to intervention. For instance, altering the rate at which people return from poor to good health may increase or decrease HLE depending on the parameterization chosen. That is a big problem. This discrepancy was pointed out by Riffe [2021, 2022] on the basis of decompositions using the linear integral decomposition approach of Horiuchi et al. [2008], but further insights were hindered by a lack of analytic treatment. In this paper we use the general formulas derived by González-Forero [2024] for the sensitivity of a recurrence to describe and treat this problem analytically for a simple multistate model.

We begin with the basic setup by presenting three parameterizations for the state transition probabilities that can be used to calculate multistate HLE. We then show how the life table response experiment decomposition method [Caswell, 1989] works with these three parameter cases, each of which implies different sensitivity equations. We then discuss selected aspects of symmetry between the sensitivities of these three parameterizations and how to transform between them.

To illustrate these findings, we use the estimates of transition probabilities recalculated from Lievre et al. [2003] to calculate the sensitivities for each of the three parameterizations and then decompose the sex gap in HLE. On the basis of this application, we argue for the use of a particular one of our three parameterizations when decomposing HLE. Specifically, we argue that HLE decompositions yield more intuitive interpretations when transition parameters are limited to forces of attrition, i.e., excluding within-state transitions.

Three parameterizations to calculate healthy life expectancy

Consider a population where individuals are in one of two health states, namely full health (h) or reduced health (u), or a single absorbing state of death (d). A life trajectory can be summarized as a sequence of states in successive ages, $Z(a)$. Denote the probability that an individual transitions from state i and age a to state j in age $a+1$ as $P(Z(a+1)=j|Z(a)=i)$ or in a simplified form as $p_{ij}(a)$. Let $\ell_i(a)$ be the fraction of individuals that have survived from age 0 to age a and are in state i at age a. Say we know or compute the initial composition (or radix) as $\ell_i(0)=P(Z(0)=i)$, where $\sum_i \ell_i(0)=1$. In this basic multistate model, the set of all possible transition probabilities in a given age a includes:

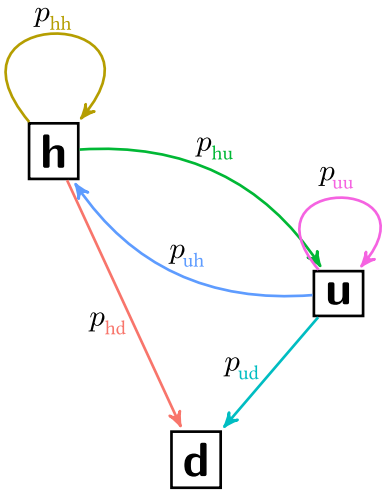
$p_{hh}(a)$: remain in full health

$p_{hu}(a)$: move from full to reduced health

$p_{hd}(a)$: move from full health to death

$p_{uh}(a)$: move from reduced to full health

Figure 1: State space diagram for the discrete time health model considered.



$p_{uu}(a)$: remain in reduced health

$p_{ud}(a)$: move from reduced health to death,

and these can be displayed in a diagram like in Fig. 1.

A hypothetical individual in this model necessarily moves from a transient health state (h, u) to one of the three possible states (h, u, d), such that the possible transitions from a given origin state sum to one:

$$p_{hd}(a)+p_{hu}(a)+p_{hh}(a)=1 \quad (1a)$$
$$p_{ud}(a)+p_{uh}(a)+p_{uu}(a)=1. \quad (1b)$$

Given the radix composition $\ell_i(0)$, one can compute the state-specific survivorship $\ell_i(a+1)$ at the next age using:

$$\ell_h(a+1)=\ell_h(a)p_{hh}(a)+\ell_u(a)p_{uh}(a) \quad (2a)$$
$$\ell_u(a+1)=\ell_u(a)p_{uu}(a)+\ell_h(a)p_{hu}(a), \quad (2b)$$

where $\ell_h(a)$ is the fraction of survivors in good health in the a^{th} age group, and $\ell_u(a)$ is the fraction of survivors in poor health. HLE is then approximately given by the marginal sum of ℓ_h over age, and its unhealthy counterpart ULE is approximated in like form:

$$\text{HLE}=\sum_a \ell_h(a) \quad (3a)$$
$$\text{ULE}=\sum_a \ell_u(a). \quad (3b)$$

Eq. (3a) is an upward-biased approximation of HLE, with an error usually around half an age interval, which we ignore in the following, although one could add precision using the approach of Schneider et al. [2023].

The first parameterization (parameterization 1, hereinafter P1) given in Eq. (2) is the most common way to calculate HLE, or rather its matrix algebra equivalent

$$\begin{pmatrix}\ell_h(a+1)\\ \ell_u(a+1)\end{pmatrix}=\begin{pmatrix}p_{hh}(a) & p_{uh}(a)\\ p_{hu}(a) & p_{uu}(a)\end{pmatrix}\begin{pmatrix}\ell_h(a)\\ \ell_u(a)\end{pmatrix} \quad (P1)$$

as described by Caswell and van Daalen [2021]. This formulation uses only transitions between and within health states (p_{hh}, p_{hu}, p_{uh}, p_{uu}), and it does not use death transitions directly. One could compute the same values of ℓ_h and ℓ_u in other ways, however. Parameterization 2 (P2) shown in Eq. (4) is based only on transitions capturing "attrition", as it lacks p_{hh} and p_{uu}:

$$\ell_h(a+1) = \ell_h(a)\left(1 - p_{hu}(a) - p_{hd}(a)\right) + \ell_u(a)\,p_{uh}(a) \tag{4a}$$

$$\ell_u(a+1) = \ell_u(a)\left(1 - p_{uh}(a) - p_{ud}(a)\right) + \ell_h(a)\,p_{hu}(a) \tag{4b}$$

or, equivalently,

$$\begin{pmatrix}\ell_h(a+1)\\ \ell_u(a+1)\end{pmatrix} = \begin{pmatrix}1 - p_{hu}(a) - p_{hd}(a) & p_{uh}(a)\\ p_{hu}(a) & 1 - p_{uh}(a) - p_{ud}(a)\end{pmatrix}\begin{pmatrix}\ell_h(a)\\ \ell_u(a)\end{pmatrix} \tag{P2}$$

Parameterization 3 (P3), per Eq.(5), is based on all transitions except those between different health states:

$$\ell_h(a+1) = \ell_h(a)\,p_{hh}(a) + \ell_u(a)\left(1 - p_{ud}(a) - p_{uu}(a)\right) \tag{5a}$$

$$\ell_u(a+1) = \ell_u(a)\,p_{uu}(a) + \ell_h(a)\left(1 - p_{hd}(a) - p_{hh}(a)\right) \tag{5b}$$

or, equivalently,

$$\begin{pmatrix}\ell_h(a+1)\\ \ell_u(a+1)\end{pmatrix} = \begin{pmatrix}p_{hh}(a) & 1 - p_{ud}(a) - p_{uu}(a)\\ 1 - p_{hd}(a) - p_{hh}(a) & p_{uu}(a)\end{pmatrix}\begin{pmatrix}\ell_h(a)\\ \ell_u(a)\end{pmatrix} \tag{P3}$$

We believe P3 has never been used in the literature before. All three parameterizations produce identical output for ℓ_i, and by extension HLE, ULE, and total life expectancy.

The problem

For measures such as HLE, it does not matter whether calculations are done using the transition matrices of P1, P2, or P3. However, if we wish to decompose differences in HLE into element-wise contributions from each transition parameter, e.g. using one of the generalized decomposition approaches [Caswell, 1989, Andreev et al., 2002, Horiuchi et al., 2008], results vary considerably depending on which parameterization is used. This inconsistency has never been recognized in the literature, and we try to thoroughly describe it in this paper. Reliable and consistent decomposition results are of utmost importance if demographic decomposition is to be used as a means to identify priorities among interventions designed to modify health transitions and mortality. Although P1 will appear to many as the most straightforward choice, we later give substantive observations that serve as a warning against this framing, and that lead us to recommend P2 specifically for purposes of decomposition.

The sensitivity of survivorship

We compute the sensitivity of survivorship using formulas for the sensitivity of a recurrence derived by González-Forero [2024], which we briefly describe in the following.

General notation

Let us denote the vector of transition probabilities included as P1 as

$$\boldsymbol{\rho}_a^{(\mathrm{P1})} = \begin{pmatrix}p_{hh}(a)\\ p_{uh}(a)\\ p_{uu}(a)\\ p_{hu}(a)\end{pmatrix} \in \mathbb{R}^4, \tag{6a}$$

in P2 as

$$\boldsymbol{\rho}_a^{(\mathrm{P2})} = \begin{pmatrix}p_{hd}(a)\\ p_{uh}(a)\\ p_{ud}(a)\\ p_{hu}(a)\end{pmatrix} \in \mathbb{R}^4, \tag{6b}$$

and in P3 as

$$\boldsymbol{\rho}_a^{(\mathrm{P3})} = \begin{pmatrix}p_{hh}(a)\\ p_{uu}(a)\\ p_{ud}(a)\\ p_{hd}(a)\end{pmatrix} \in \mathbb{R}^4. \tag{6c}$$

Thus, we can define, more generally, the vector of transition probabilities as

$$\boldsymbol{\rho}_a^{(c)} = \begin{pmatrix}\rho_1^{(c)}(a)\\ \rho_2^{(c)}(a)\\ \rho_3^{(c)}(a)\\ \rho_4^{(c)}(a)\end{pmatrix} \in \mathbb{R}^4, \tag{6d}$$

with respective entries depending on the parameterization $c \in \{\mathrm{P1}, \mathrm{P2}, \mathrm{P3}\}$ at age $a \in \{0, \ldots, n\}$ (e.g., $\rho_3^{(\mathrm{P2})}(a) = p_{ud}(a)$). Consider the matrix $\mathbf{P}$ whose a-th column is $\boldsymbol{\rho}_a^{(c)}$. Then, let us define the column vector of transition probabilities used in parameterization c over all ages as $\boldsymbol{\rho}^{(c)} = \mathrm{vec}(\mathbf{P}) \in \mathbb{R}^{4(n+1)}$.

Let $\boldsymbol{\ell}_a = \left(\ell_h(a), \ell_u(a)\right)^\intercal \in \mathbb{R}^2$ be the (column) vector of survivorship for the two health states at age a. Then, for each $c \in \{\mathrm{P1}, \mathrm{P2}, \mathrm{P3}\}$, we can succinctly write the three parameterizations in Eqs. P1–P3 as

$$\boldsymbol{\ell}_{a+1} = \mathbf{g}^{(c)}\left(\boldsymbol{\ell}_a, \boldsymbol{\rho}_a^{(c)}\right), \tag{7}$$

where function $\mathbf{g}^{(c)}(\cdot)$ is given by the right-hand side of Eqs. P1–P3, respectively.

Equation (7) indicates that survivorship to age $a+1$ depends both on survivorship and transition probabilities at the immediately preceding age. We will thus compute the sensitivity of survivorship to perturbations of both survivorship and transition probabilities at any previous age. Note that, while the transition probabilities $\boldsymbol{\rho}_a^{(c)}$ and the function $\mathbf{g}^{(c)}$ depend explicitly on the parameterization c, the survivorship $\boldsymbol{\ell}_{a+1}$ does not. However, we will show that the sensitivity of survivorship does actually depend on the parameterization.

To denote derivatives of vectors with respect to vectors, we adopt the notation used by Caswell [2019] from matrix calculus. For column vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, we denote the partial derivative of $\mathbf{x}^\top$ with respect to $\mathbf{y}$ as

$$\frac{\partial \mathbf{x}^\top}{\partial \mathbf{y}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial y_m} & \cdots & \frac{\partial x_n}{\partial y_m} \end{pmatrix} \in \mathbb{R}^{m\times n}.$$

The sensitivity of survivorship to perturbations in survivorship

We now construct the formulas for the sensitivity of survivorship to perturbations in survivorship (at earlier ages) that apply to all three parameterizations by applying the formulas for the sensitivity of a recurrence derived by González-Forero [2024]. The direct effects on survivorship at age $a+1$ ($\boldsymbol{\ell}_{a+1}$) of perturbing survivorship at the immediately preceding age ($\boldsymbol{\ell}_a$) are given by the matrix

$$\frac{\partial \boldsymbol{\ell}_{a+1}^\top}{\partial \boldsymbol{\ell}_a} = \begin{pmatrix} \frac{\partial \ell_\mathrm{h}(a+1)}{\partial \ell_\mathrm{h}(a)} & \frac{\partial \ell_\mathrm{u}(a+1)}{\partial \ell_\mathrm{h}(a)} \\ \frac{\partial \ell_\mathrm{h}(a+1)}{\partial \ell_\mathrm{u}(a)} & \frac{\partial \ell_\mathrm{u}(a+1)}{\partial \ell_\mathrm{u}(a)} \end{pmatrix} \in \mathbb{R}^{2\times 2} \tag{8}$$

(from Eq. 19 in González-Forero [2024]).

Consider the matrix $\mathbf{L}$ whose a-th column is $\boldsymbol{\ell}_a$. Let us define $\boldsymbol{\ell} = \mathrm{vec}(\mathbf{L}) \in \mathbb{R}^{2(n+1)}$ as the column vector of survivorship for the two health states for all ages $a \in \{0,\ldots,n\}$. The direct effects on survivorship at any age of perturbing survivorship at any other age are given by the matrix

$$\begin{aligned} \frac{\partial \boldsymbol{\ell}^\top}{\partial \boldsymbol{\ell}} &= \begin{pmatrix} \frac{\partial \boldsymbol{\ell}_0^\top}{\partial \boldsymbol{\ell}_0} & \cdots & \frac{\partial \boldsymbol{\ell}_a^\top}{\partial \boldsymbol{\ell}_0} \\ \vdots & \ddots & \vdots \\ \frac{\partial \boldsymbol{\ell}_0^\top}{\partial \boldsymbol{\ell}_a} & \cdots & \frac{\partial \boldsymbol{\ell}_n^\top}{\partial \boldsymbol{\ell}_n} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{I} & \frac{\partial \boldsymbol{\ell}_1^\top}{\partial \boldsymbol{\ell}_0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \frac{\partial \boldsymbol{\ell}_n^\top}{\partial \boldsymbol{\ell}_{n-1}} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} \end{pmatrix} \\ &\in \mathbb{R}^{2(n+1)\times 2(n+1)}, \end{aligned} \tag{9}$$

(from Eq. 18 of González-Forero [2024]). Note that each of the elements of the second matrix in Eq. (9) is a block of dimension 2×2. The identity and zero block entries arise because perturbing a variable at a given age can only directly affect itself or another variable at the immediately next age.

The sensitivity of survivorship is given by the total effects on survivorship of a perturbation, which are the accumulation of direct effects over life of the initial perturbation and indirect effects accumulated in later ages. Whereas direct effects are given by partial derivatives, denoted by "∂", total effects are given by total derivatives, denoted by "d". Thus, the sensitivity of survivorship at age a' to a perturbation in survivorship at age a is given by

$$\frac{\mathrm{d}\boldsymbol{\ell}_{a'}^\top}{\mathrm{d}\boldsymbol{\ell}_a} = \begin{cases} \prod_{j=a}^{\curvearrowleft a'-1} \frac{\partial \boldsymbol{\ell}_{j+1}^\top}{\partial \boldsymbol{\ell}_j} = \frac{\partial \boldsymbol{\ell}_{a+1}^\top}{\partial \boldsymbol{\ell}_a} \cdots \frac{\partial \boldsymbol{\ell}_{a'}^\top}{\partial \boldsymbol{\ell}_{a'-1}} & \text{for } a' > a \\ \mathbf{I} & \text{for } a' = a \\ \mathbf{0} & \text{for } a' < a \end{cases}. \tag{10}$$

(from Eq. 12 in González-Forero [2024]). The arrow $\curvearrowright$ denotes right multiplication. In this equation, for $a' > a$ we have the direct effects of the perturbation at age a on the survivorship at the next age $a+1$ multiplied by the direct effects of such perturbation at age $a+1$ on the survivorship at the next age, and so on all the way to age a'. That is, the total effects of the perturbation at age a on survivorship at age a' are the accumulation of direct effects of the initial perturbation and unleashed direct effects.

In particular, the sensitivity of survivorship at age $a > 0$ to an independent perturbation in the initial conditions is given by

$$\frac{\mathrm{d}\boldsymbol{\ell}_a^\top}{\mathrm{d}\boldsymbol{\ell}_0} = \frac{\partial \boldsymbol{\ell}_1^\top}{\partial \boldsymbol{\ell}_0} \cdots \frac{\partial \boldsymbol{\ell}_a^\top}{\partial \boldsymbol{\ell}_{a-1}}. \tag{11}$$

The qualifier "independent" here means that the sensitivity in Eq. (11) perturbs, for instance, $\ell_\mathrm{h}(0)$ keeping $\ell_\mathrm{u}(0)$ constant, and vice-versa, without considering the constraint $\ell_\mathrm{h}(0) + \ell_\mathrm{u}(0) = 1$.

The sensitivity in Eq. (10) can be more succinctly computed as the a-th block row and a'-th block column of

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{\ell}^\top}{\mathrm{d}\boldsymbol{\ell}} &= \left(2\mathbf{I} - \frac{\partial \boldsymbol{\ell}^\top}{\partial \boldsymbol{\ell}}\right)^{-1} \\ &= \begin{pmatrix} \mathbf{I} & \frac{\mathrm{d}\boldsymbol{\ell}_1^\top}{\mathrm{d}\boldsymbol{\ell}_0} & \cdots & \frac{\mathrm{d}\boldsymbol{\ell}_{n-1}^\top}{\mathrm{d}\boldsymbol{\ell}_0} & \frac{\mathrm{d}\boldsymbol{\ell}_n^\top}{\mathrm{d}\boldsymbol{\ell}_0} \\ \mathbf{0} & \mathbf{I} & \cdots & \frac{\mathrm{d}\boldsymbol{\ell}_{n-1}^\top}{\mathrm{d}\boldsymbol{\ell}_1} & \frac{\mathrm{d}\boldsymbol{\ell}_n^\top}{\mathrm{d}\boldsymbol{\ell}_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \frac{\mathrm{d}\boldsymbol{\ell}_n^\top}{\mathrm{d}\boldsymbol{\ell}_{n-1}} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} \end{pmatrix} \end{aligned} \tag{12}$$

(from Eq. 11 and Eq. S5.1.15 in González-Forero [2024]). In particular, the sensitivity of survivorship in later ages to independent changes in the initial conditions (i.e. the fraction in each state at the first time step) is found in the first block row of the matrix defined in (12). This matrix of total effects of survivorship on itself describes the survivorship feedback that occurs over life. This matrix recovers values of the fundamental matrix of a Markov chain when the function $\mathbf{g}$ in Eq. (7) is linear with respect to $\boldsymbol{\ell}_a$, such that Eq. (8) gives the matrix of transitions (see section 5.1.2.1 in Caswell [2001]).

Let $\mathbf{v} = (\mathrm{HLE}, \mathrm{ULE})^\top \in \mathbb{R}^2$ be the vector of healthy and unhealthy life expectancy. Its sensitivity to independent perturbations in the initial conditions is given by

$$\frac{\mathrm{d}\mathbf{v}^\top}{\mathrm{d}\boldsymbol{\ell}_0} = \sum_{a=0}^{n} \frac{\mathrm{d}\boldsymbol{\ell}_a^\top}{\mathrm{d}\boldsymbol{\ell}_0} = \begin{pmatrix} \frac{\mathrm{dHLE}}{\mathrm{d}\ell_\mathrm{h}(0)} & \frac{\mathrm{dULE}}{\mathrm{d}\ell_\mathrm{h}(0)} \\ \frac{\mathrm{dHLE}}{\mathrm{d}\ell_\mathrm{u}(0)} & \frac{\mathrm{dULE}}{\mathrm{d}\ell_\mathrm{u}(0)} \end{pmatrix}, \tag{13}$$

which is the sum of the top block row in Eq. (12).

To compute the sensitivity of HLE to a perturbation in the initial conditions, we must consider the necessary simultaneous perturbation of $\ell_{\mathrm{h}}(0)$ and $\ell_{\mathrm{u}}(0)$, constrained to $\ell_{\mathrm{h}}(0)+\ell_{\mathrm{u}}(0)=1$. We can express HLE as a function of the initial conditions $\ell_{\mathrm{h}}(0)$ and $\ell_{\mathrm{u}}(0)$. Substituting for the constraint, we have $\mathrm{HLE}=\mathrm{HLE}(\ell_{\mathrm{h}}(0),1-\ell_{\mathrm{h}}(0))$. Hence, using the chain rule, the sensitivity of HLE to perturbation in the initial $\ell_{\mathrm{h}}(0)$ is

$$\begin{aligned}\frac{\mathbb{d}\mathrm{HLE}}{\mathbb{d}\ell_{\mathrm{h}}(0)} &= \frac{\mathrm{dHLE}}{\mathrm{d}\ell_{\mathrm{h}}(0)}\frac{\mathrm{d}\ell_{\mathrm{h}}(0)}{\mathrm{d}\ell_{\mathrm{h}}(0)} + \frac{\mathrm{dHLE}}{\mathrm{d}\ell_{\mathrm{u}}(0)}\frac{\mathrm{d}(1-\ell_{\mathrm{h}}(0))}{\mathbb{d}\ell_{\mathrm{h}}(0)} \\ &= \frac{\mathrm{dHLE}}{\mathrm{d}\ell_{\mathrm{h}}(0)} - \frac{\mathrm{dHLE}}{\mathrm{d}\ell_{\mathrm{u}}(0)} \end{aligned} \tag{14}$$

where we use the symbol $\mathbb{d}$ to denote a total differential considering the initial condition constraint, to distinguish it from the total differential d that ignores that constraint.

The sensitivity of survivorship to perturbations in transition probabilities

We can similarly construct the formulas for the sensitivity of survivorship to perturbations in transition probabilities that apply for all three parameterizations. The direct effects on survivorship at age $a+1$ of perturbing state-transition probabilities at the immediately preceding age a are given by the matrix

$$\frac{\partial \boldsymbol{\ell}_{a+1}^{\mathsf{T}}}{\partial \boldsymbol{\rho}_a^{(c)}} = \begin{pmatrix} \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho_1^{(c)}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_1^{(c)}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho_2^{(c)}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_2^{(c)}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho_3^{(c)}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_3^{(c)}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho_4^{(c)}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_4^{(c)}(a)} \end{pmatrix} \in \mathbb{R}^{4\times 2} \tag{15}$$

(from the equation following Eq. S5.2.7 in González-Forero [2024]). With this, the direct effects on survivorship at any age of perturbing transition probabilities at any other age are given by the matrix

$$\begin{aligned}\frac{\partial \boldsymbol{\ell}^{\mathsf{T}}}{\partial \boldsymbol{\rho}^{(c)}} &= \begin{pmatrix} \frac{\partial \boldsymbol{\ell}_0^{\mathsf{T}}}{\partial \boldsymbol{\rho}_0^{(c)}} & \cdots & \frac{\partial \boldsymbol{\ell}_n^{\mathsf{T}}}{\partial \boldsymbol{\rho}_0^{(c)}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \boldsymbol{\ell}_0^{\mathsf{T}}}{\partial \boldsymbol{\rho}_n^{(c)}} & \cdots & \frac{\partial \boldsymbol{\ell}_n^{\mathsf{T}}}{\partial \boldsymbol{\rho}_n^{(c)}} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0} & \frac{\partial \boldsymbol{\ell}_1^{\mathsf{T}}}{\partial \boldsymbol{\rho}_0^{(c)}} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \frac{\partial \boldsymbol{\ell}_n^{\mathsf{T}}}{\partial \boldsymbol{\rho}_{n-1}^{(c)}} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{pmatrix} \\ &\in \mathbb{R}^{4(n+1)\times 2(n+1)} \end{aligned} \tag{16}$$

(from Layer 2, Eq. S2b in González-Forero [2024]).

As the sensitivity of survivorship is the accumulation of direct effects over life, the sensitivity of survivorship at age j to a perturbation in survivorship at age $a<j$ is given by

$$\frac{\mathrm{d}\boldsymbol{\ell}_j^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\rho}_a^{(c)}} = \frac{\partial \boldsymbol{\ell}_{a+1}^{\mathsf{T}}}{\partial \boldsymbol{\rho}_a^{(c)}}\frac{\mathrm{d}\boldsymbol{\ell}_j^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\ell}_{a+1}} \tag{17}$$

(from Eq. S5.2.17 in González-Forero [2024]), where the first derivative on the right-hand side is given by Eq. (15) and the second is given by Eq. (10). The sensitivity in Eq. (17) can be more succinctly computed as the a-th block row and a'-th block column of

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{\ell}^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\rho}^{(c)}} &= \frac{\partial \boldsymbol{\ell}^{\mathsf{T}}}{\partial \boldsymbol{\rho}^{(c)}}\frac{\mathrm{d}\boldsymbol{\ell}^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\ell}} \\ &= \begin{pmatrix} \frac{\mathrm{d}\boldsymbol{\ell}_0^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\rho}_0^{(c)}} & \cdots & \frac{\mathrm{d}\boldsymbol{\ell}_n^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\rho}_0^{(c)}} \\ \vdots & \ddots & \vdots \\ \frac{\mathrm{d}\boldsymbol{\ell}_0^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\rho}_n^{(c)}} & \cdots & \frac{\mathrm{d}\boldsymbol{\ell}_n^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\rho}_n^{(c)}} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0} & \frac{\mathrm{d}\boldsymbol{\ell}_1^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\rho}_0^{(c)}} & \cdots & \frac{\mathrm{d}\boldsymbol{\ell}_{n-1}^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\rho}_0^{(c)}} & \frac{\mathrm{d}\boldsymbol{\ell}_n^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\rho}_0^{(c)}} \\ \mathbf{0} & \mathbf{0} & \cdots & \frac{\mathrm{d}\boldsymbol{\ell}_{n-1}^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\rho}_1^{(c)}} & \frac{\mathrm{d}\boldsymbol{\ell}_n^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\rho}_1^{(c)}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \frac{\mathrm{d}\boldsymbol{\ell}_n^{\mathsf{T}}}{\mathrm{d}\boldsymbol{\rho}_{n-1}^{(c)}} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{aligned} \tag{18}$$

(from Eq. 10 and Eq. S5.2.16 in González-Forero [2024]).

The sensitivity of HLE

Since $\mathrm{HLE}=\sum_{a=0}^{n}\ell_{\mathrm{h}}(a)$, the sensitivity of HLE to perturbation in the i-th transition probability $\rho_i^{(c)}(a)$ of parameterization c at age a is ($i\in\{1,\ldots,4\}$):

$$s_{\mathrm{h},\sigma(i,c),a}^{(c)} = \frac{\mathrm{dHLE}}{\mathrm{d}\rho_i^{(c)}(a)} = \sum_{j=a}^{n}\frac{\mathrm{d}\ell_{\mathrm{h}}(j)}{\mathrm{d}\rho_i^{(c)}(a)}, \tag{19}$$

where σ is a function such that $\sigma(i,c)$ gives the input and output states for the i-th entry of $\boldsymbol{\rho}_a$ under parameterization c (e.g., hh; Eq. (6)). We can write this equation in a form that separates the factors that depend on the parameterization. Substituting Eq. (17) evaluated at the i-th entry of $\boldsymbol{\rho}_a^{(c)}$ and the h entry of $\boldsymbol{\ell}_j^{\mathsf{T}}$ into Eq. (19) yields

$$s_{\mathrm{h},\sigma(i,c),a}^{(c)} = \sum_{j=a}^{n}\frac{\partial \boldsymbol{\ell}_{a+1}^{\mathsf{T}}}{\partial \rho_i^{(c)}(a)}\frac{\mathrm{d}\ell_{\mathrm{h}}(j)}{\mathrm{d}\boldsymbol{\ell}_{a+1}}.$$

Taking the factor that is independent of j outside of the sum yields

$$s_{\mathrm{h},\sigma(i,c),a}^{(c)} = \frac{\partial \boldsymbol{\ell}_{a+1}^{\mathsf{T}}}{\partial \rho_{ia}^{(c)}}\sum_{j=a}^{n}\frac{\mathrm{d}\ell_{\mathrm{h}}(j)}{\mathrm{d}\boldsymbol{\ell}_{a+1}}.$$

Recalling that $\boldsymbol{\ell}_a = (\ell_{\mathrm{h}}(a); \ell_{\mathrm{u}}(a))$ and expanding the indicated matrix multiplication yields

$$s^{(c)}_{\mathrm{h},\sigma(i,c),a} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho^{(c)}_i(a)} \sum_{j=a}^{n} \frac{\mathrm{d}\ell_{\mathrm{h}}(j)}{\mathrm{d}\ell_{\mathrm{h}}(a+1)} + \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho^{(c)}_i(a)} \sum_{j=a}^{n} \frac{\mathrm{d}\ell_{\mathrm{h}}(j)}{\mathrm{d}\ell_{\mathrm{u}}(a+1)}.$$

Denoting the two sums by A and B, respectively, we have

$$s^{(c)}_{\mathrm{h},\sigma(i,c),a} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho^{(c)}_i(a)} A + \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho^{(c)}_i(a)} B. \tag{20}$$

The quantities A and B do not depend on the parameterization because they only depend on the direct effects of states (Eq. (12)), which we will see are independent of the parameterization.

The vector of sensitivities of HLE to perturbation in transition probabilities at age a for parameterization c is

$$\mathbf{s}^{(c)}_{\mathrm{h}a} = \frac{\mathrm{dHLE}}{\mathrm{d}\boldsymbol{\rho}^{(c)}_a} = \sum_{j=a}^{n} \frac{\mathrm{d}\ell_{\mathrm{h}}(j)}{\mathrm{d}\boldsymbol{\rho}^{(c)}_a}. \tag{21}$$

and over all ages is

$$\mathbf{s}^{(c)}_{\mathrm{h}} = \frac{\mathrm{dHLE}}{\mathrm{d}\boldsymbol{\rho}^{(c)}} = \sum_{j=a}^{n} \frac{\mathrm{d}\ell_{\mathrm{h}}(j)}{\mathrm{d}\boldsymbol{\rho}^{(c)}}. \tag{22}$$

Similarly, the sensitivity of HLE to independent perturbation in the initial conditions is given by

$$\frac{\mathrm{dHLE}}{\mathrm{d}\boldsymbol{\ell}_0} = \sum_{j=0}^{n} \frac{\mathrm{d}\ell_{\mathrm{h}}(j)}{\mathrm{d}\boldsymbol{\ell}_0}. \tag{23}$$

which is the left column of the 2×2 matrix given in Eq. (13).

The decomposition of HLE

Following Caswell's decomposition of the effect of a treatment on a vital rate ([Caswell, 1989], Caswell 2001, p. 261), we compute the contributions of marginally perturbed transitions to a change in HLE as follows: Consider two vectors of transition probabilities, $\boldsymbol{\rho}^{(c)}$ and $\boldsymbol{\rho}^{(c)'}$, and let HLE and HLE′ be the healthy life expectancies that arise under each of them. $\boldsymbol{\rho}^{(c)}$ and $\boldsymbol{\rho}^{(c)'}$ might be the health and mortality transitions of two distinct populations, or the same populations at two points in time, or an observed population versus a hypothetical scenario. The first order approximation of HLE′ with respect to $\boldsymbol{\rho}^{(c)'}$ around $\boldsymbol{\rho}^{(c)}$ is

$$\begin{aligned}
\mathrm{HLE}' &\approx \mathrm{HLE} + \left.\frac{\mathrm{dHLE}}{\mathrm{d}\boldsymbol{\rho}^{(c)\mathsf{T}}}\right|_{\bar{\boldsymbol{\rho}}} (\boldsymbol{\rho}^{(c)'} - \boldsymbol{\rho}^{(c)}) \\
&= \mathrm{HLE} + \sum_{k=1}^{4} \sum_{j=0}^{n} \left.\frac{\mathrm{dHLE}}{\mathrm{d}\rho^{(c)}_k(j)}\right|_{\bar{\boldsymbol{\rho}}} (\rho^{(c)'}_k(j) - \rho^{(c)}_k(j)) \\
&= \mathrm{HLE} + \sum_{k=1}^{4} \sum_{j=0}^{n} \sum_{i=0}^{n} \left.\frac{\mathrm{d}\ell_{\mathrm{h}}(i)}{\mathrm{d}\rho^{(c)}_k(j)}\right|_{\bar{\boldsymbol{\rho}}} (\rho^{(c)'}_k(j) - \rho^{(c)}_k(j)),
\end{aligned} \tag{24}$$

where the derivative $\mathrm{d}\ell_{\mathrm{h}}(i)/\mathrm{d}\rho^{(c)}_k(j)$ necessarily evaluates to zero in ages $i < j$ due to the upper triangular structure of the matrix Eq. (18). The derivatives are conventionally evaluated at

$$\bar{\boldsymbol{\rho}} = \frac{1}{2}\left(\boldsymbol{\rho}^{(c)'} + \boldsymbol{\rho}^{(c)}\right) \tag{25}$$

to improve the approximation (Caswell 2001, p. 261-262). Each kj-th term in the sums in Eq. (24) gives the contribution of the k-th transition probability at age j to the resulting differences in HLE.

Hence, denoting $\boldsymbol{\delta} = \boldsymbol{\rho}^{(c)'} - \boldsymbol{\rho}^{(c)}$ and $\Delta = \mathrm{HLE}' - \mathrm{HLE}$, we have

$$\Delta \approx \boldsymbol{\delta}^{\mathsf{T}} \mathbf{s}^{(c)}_{\mathrm{h}}|_{\bar{\mathbf{p}}}, \tag{26}$$

which is the dot product between the vectors $\boldsymbol{\delta}$ and $\mathbf{s}^{(c)}_{\mathrm{h}}|_{\bar{\mathbf{p}}}$ returning the scalar Δ. Since in Eq. (26), both the sensitivity vector and perturbation vector depend on the parameterization c as we will see, the effect on HLE of changing state transitions depends on the parameterization.

Parameterization 1

We now compute the matrices of direct effects for P1. The direct effects of the states on themselves under P1 are (Eq. (8)):

$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \ell_{\mathrm{h}}(a)} = p_{\mathrm{hh}}(a) \tag{27a}$$
$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \ell_{\mathrm{u}}(a)} = p_{\mathrm{uh}}(a) \tag{27b}$$
$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \ell_{\mathrm{u}}(a)} = p_{\mathrm{uu}}(a) \tag{27c}$$
$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \ell_{\mathrm{h}}(a)} = p_{\mathrm{hu}}(a). \tag{27d}$$

We then have the matrix of direct effects of states at age a:

$$\frac{\partial \boldsymbol{\ell}^{\mathsf{T}}_{a+1}}{\partial \boldsymbol{\ell}_a} = \begin{pmatrix} \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \ell_{\mathrm{h}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \ell_{\mathrm{h}}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \ell_{\mathrm{u}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \ell_{\mathrm{u}}(a)} \end{pmatrix} = \begin{pmatrix} p_{\mathrm{hh}}(a) & p_{\mathrm{hu}}(a) \\ p_{\mathrm{uh}}(a) & p_{\mathrm{uu}}(a). \end{pmatrix}. \tag{28}$$

which equals the transition matrix due to the linearity of $\mathbf{g}^{(c)}$ with respect to states in Eq. (7).

Similarly, the direct effects of the transitions on healthy years lived in the next age, $\ell_{\mathrm{h}}(a+1)$, are (Eq. (15) under P1):

$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho^{(\mathrm{P1})}_1(a)} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{hh}}(a)} = \ell_{\mathrm{h}}(a) \tag{29a}$$
$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho^{(\mathrm{P1})}_2(a)} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{uh}}(a)} = \ell_{\mathrm{u}}(a) \tag{29b}$$
$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho^{(\mathrm{P1})}_3(a)} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{uu}}(a)} = 0 \tag{29c}$$
$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho^{(\mathrm{P1})}_4(a)} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{hu}}(a)} = 0, \tag{29d}$$

and the direct effects of the transitions on the reduced

health state are:

$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_1^{(\mathrm{P1})}(a)} = \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{hh}}(a)} = 0 \tag{30a}$$
$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_2^{(\mathrm{P1})}(a)} = \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{uh}}(a)} = 0 \tag{30b}$$
$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_3^{(\mathrm{P1})}(a)} = \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{uu}}(a)} = \ell_{\mathrm{u}}(a) \tag{30c}$$
$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_4^{(\mathrm{P1})}(a)} = \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{hu}}(a)} = \ell_{\mathrm{h}}(a). \tag{30d}$$

We then have the matrix of direct effects of the transitions on the states at a

$$\frac{\partial \boldsymbol{\ell}_{a+1}^{\top}}{\partial \boldsymbol{\rho}_a^{(\mathrm{P1})}} = \begin{pmatrix} \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{hh}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{hh}}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{uh}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{uh}}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{uu}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{uu}}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{hu}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{hu}}(a)} \end{pmatrix} = \begin{pmatrix} \ell_{\mathrm{h}}(a) & 0 \\ \ell_{\mathrm{u}}(a) & 0 \\ 0 & \ell_{\mathrm{u}}(a) \\ 0 & \ell_{\mathrm{h}}(a) \end{pmatrix}. \tag{31}$$

Parameterization 2

We now compute the matrices of direct effects for P2.

The direct effects of the states on themselves under P2 are (Eq. (8)):

$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \ell_{\mathrm{h}}(a)} = 1 - p_{\mathrm{hd}}(a) - p_{\mathrm{hu}}(a) \tag{32a}$$
$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \ell_{\mathrm{u}}(a)} = p_{\mathrm{uh}}(a) \tag{32b}$$
$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \ell_{\mathrm{h}}(a)} = p_{\mathrm{hu}}(a) \tag{32c}$$
$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \ell_{\mathrm{u}}(a)} = 1 - p_{\mathrm{ud}}(a) - p_{\mathrm{uh}}(a). \tag{32d}$$

We then have the matrix of direct effects of states at age a:

$$\frac{\partial \boldsymbol{\ell}_{a+1}^{\top}}{\partial \boldsymbol{\ell}_a} = \begin{pmatrix} \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \ell_{\mathrm{h}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \ell_{\mathrm{h}}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \ell_{\mathrm{u}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \ell_{\mathrm{u}}(a)} \end{pmatrix} = \begin{pmatrix} 1 - p_{\mathrm{hd}}(a) - p_{\mathrm{hu}}(a) & p_{\mathrm{hu}}(a) \\ p_{\mathrm{uh}}(a) & 1 - p_{\mathrm{ud}}(a) - p_{\mathrm{uh}}(a). \end{pmatrix}, \tag{33}$$

the entries of which are identical to that for P1 in Eq. (28) due to Eq. (1).

Similarly, the direct effects of the transitions on healthy years lived in the next age, $\ell_{\mathrm{h}}(a+1)$, under P2 are (Eq. (15)):

$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho_1^{(\mathrm{P2})}(a)} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{hd}}(a)} = -\ell_{\mathrm{h}}(a) \tag{34a}$$
$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho_2^{(\mathrm{P2})}(a)} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{uh}}(a)} = \ell_{\mathrm{u}}(a) \tag{34b}$$
$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho_3^{(\mathrm{P2})}(a)} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{ud}}(a)} = 0 \tag{34c}$$
$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho_4^{(\mathrm{P2})}(a)} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{hu}}(a)} = -\ell_{\mathrm{h}}(a), \tag{34d}$$

and the direct effects of the transitions on unhealthy years lived in the next age, $\ell_{\mathrm{u}}(a+1)$, are:

$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_1^{(\mathrm{P2})}(a)} = \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{hd}}(a)} = 0 \tag{35a}$$
$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_2^{(\mathrm{P2})}(a)} = \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{uh}}(a)} = -\ell_{\mathrm{u}}(a) \tag{35b}$$
$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_3^{(\mathrm{P2})}(a)} = \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{ud}}(a)} = -\ell_{\mathrm{u}}(a) \tag{35c}$$
$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_4^{(\mathrm{P2})}(a)} = \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{hu}}(a)} = \ell_{\mathrm{h}}(a). \tag{35d}$$

We then have the matrix of direct effects of the transitions on the states at age a

$$\frac{\partial \boldsymbol{\ell}_{a+1}^{\top}}{\partial \boldsymbol{\rho}_a^{(\mathrm{P2})}} = \begin{pmatrix} \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{hd}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{hd}}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{uh}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{uh}}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{ud}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{ud}}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{hu}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{hu}}(a)} \end{pmatrix} = \begin{pmatrix} -\ell_{\mathrm{h}}(a) & 0 \\ \ell_{\mathrm{u}}(a) & -\ell_{\mathrm{u}}(a) \\ 0 & -\ell_{\mathrm{u}}(a) \\ -\ell_{\mathrm{h}}(a) & \ell_{\mathrm{h}}(a) \end{pmatrix}, \tag{36}$$

which is different from that of P1 in Eq. (31).

Parameterization 3

We now compute the matrices of direct effects for P3.

The direct effects of the states on themselves for P3 under P3 are (Eq. (8)):

$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \ell_{\mathrm{h}}(a)} = p_{\mathrm{hh}}(a) \tag{37a}$$
$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \ell_{\mathrm{u}}(a)} = 1 - p_{\mathrm{ud}}(a) - p_{\mathrm{uu}}(a) \tag{37b}$$
$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \ell_{\mathrm{h}}(a)} = 1 - p_{\mathrm{hd}}(a) - p_{\mathrm{hh}}(a) \tag{37c}$$
$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \ell_{\mathrm{u}}(a)} = p_{\mathrm{uu}}(a). \tag{37d}$$

We then have the matrix of direct effects of states at a:

$$\frac{\partial \boldsymbol{\ell}_{a+1}^{\intercal}}{\partial \boldsymbol{\ell}_a} = \begin{pmatrix} \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \ell_{\mathrm{h}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \ell_{\mathrm{h}}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \ell_{\mathrm{u}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \ell_{\mathrm{u}}(a)} \end{pmatrix} \tag{38}$$

$$= \begin{pmatrix} p_{\mathrm{hh}}(a) & 1 - p_{\mathrm{hd}}(a) - p_{\mathrm{hh}}(a) \\ 1 - p_{\mathrm{ud}}(a) - p_{\mathrm{uu}}(a) & p_{\mathrm{uu}}(a). \end{pmatrix}. \tag{39}$$

whose entries are identical to those of P1 in Eq. (28) due to Eq. (1).

Similarly, the direct effects of the transitions on healthy years lived in the next age, $\ell_{\mathrm{h}}(a+1)$, under P3 are (Eq. (15)):

$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho_1^{(\mathrm{P3})}(a)} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{hh}}(a)} = \ell_{\mathrm{h}}(a) \tag{40a}$$

$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho_2^{(\mathrm{P3})}(a)} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{uu}}(a)} = -\ell_{\mathrm{u}}(a) \tag{40b}$$

$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho_3^{(\mathrm{P3})}(a)} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{ud}}(a)} = -\ell_{\mathrm{u}}(a) \tag{40c}$$

$$\frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial \rho_4^{(\mathrm{P3})}(a)} = \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{hd}}(a)} = 0, \tag{40d}$$

and the direct effects of the transitions on unhealthy years lived in the next age, $\ell_{\mathrm{u}}(a+1)$, are:

$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_1^{(\mathrm{P3})}(a)} = \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{hh}}(a)} = -\ell_{\mathrm{h}}(a) \tag{41a}$$

$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_2^{(\mathrm{P3})}(a)} = \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{uu}}(a)} = \ell_{\mathrm{u}}(a) \tag{41b}$$

$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_3^{(\mathrm{P3})}(a)} = \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{ud}}(a)} = 0 \tag{41c}$$

$$\frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial \rho_4^{(\mathrm{P3})}(a)} = \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{hd}}(a)} = -\ell_{\mathrm{h}}(a). \tag{41d}$$

We then have the matrix of direct effects of the transitions on the states at a

$$\frac{\partial \boldsymbol{\ell}_{a+1}^{\intercal}}{\partial \boldsymbol{\rho}_a^{(\mathrm{P3})}} = \begin{pmatrix} \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{hh}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{hh}}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{uu}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{uu}}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{ud}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{ud}}(a)} \\ \frac{\partial \ell_{\mathrm{h}}(a+1)}{\partial p_{\mathrm{hd}}(a)} & \frac{\partial \ell_{\mathrm{u}}(a+1)}{\partial p_{\mathrm{hd}}(a)} \end{pmatrix} = \begin{pmatrix} \ell_{\mathrm{h}}(a) & -\ell_{\mathrm{h}}(a) \\ -\ell_{\mathrm{u}}(a) & \ell_{\mathrm{u}}(a) \\ -\ell_{\mathrm{u}}(a) & 0 \\ 0 & -\ell_{\mathrm{h}}(a) \end{pmatrix}. \tag{42}$$

which is different from those of P1 in Eq. (31) and P2 in Eq. (36).

Symmetries between the sensitivities of each parameterization

From Eq. (20), many relationships between the sensitivities follow. Some examples are below.

Using the direct effects of transitions in Eq. (20), for P1 (Eq. (31)) we have the sensitivities

$$s_{\mathrm{h},\sigma(1,1),a}^{(\mathrm{P1})} = s_{\mathrm{h},\mathrm{hh},a}^{(\mathrm{P1})} = \ell_{\mathrm{h}}(a)A \tag{43a}$$

$$s_{\mathrm{h},\sigma(2,1),a}^{(\mathrm{P1})} = s_{\mathrm{h},\mathrm{uh},a}^{(\mathrm{P1})} = \ell_{\mathrm{u}}(a)A \tag{43b}$$

$$s_{\mathrm{h},\sigma(3,1),a}^{(\mathrm{P1})} = s_{\mathrm{h},\mathrm{uu},a}^{(\mathrm{P1})} = \ell_{\mathrm{u}}(a)B \tag{43c}$$

$$s_{\mathrm{h},\sigma(4,1),a}^{(\mathrm{P1})} = s_{\mathrm{h},\mathrm{hu},a}^{(\mathrm{P1})} = \ell_{\mathrm{h}}(a)B. \tag{43d}$$

For P2 (Eq. (36)), we have the sensitivities

$$s_{\mathrm{h},\sigma(1,2),a}^{(\mathrm{P2})} = s_{\mathrm{h},\mathrm{hd},a}^{(\mathrm{P2})} = -\ell_{\mathrm{h}}(a)A \tag{44a}$$

$$s_{\mathrm{h},\sigma(2,2),a}^{(\mathrm{P2})} = s_{\mathrm{h},\mathrm{uh},a}^{(\mathrm{P2})} = \ell_{\mathrm{u}}(a)A - \ell_{\mathrm{u}}(a)B \tag{44b}$$

$$s_{\mathrm{h},\sigma(3,2),a}^{(\mathrm{P2})} = s_{\mathrm{h},\mathrm{ud},a}^{(\mathrm{P2})} = -\ell_{\mathrm{u}}(a)B \tag{44c}$$

$$s_{\mathrm{h},\sigma(4,2),a}^{(\mathrm{P2})} = s_{\mathrm{h},\mathrm{hu},a}^{(\mathrm{P2})} = -\ell_{\mathrm{h}}(a)A + \ell_{\mathrm{h}}(a)B. \tag{44d}$$

For P3 (Eq. (42)), we have the sensitivities

$$s_{\mathrm{h},\sigma(1,3),a}^{(\mathrm{P3})} = s_{\mathrm{h},\mathrm{hh},a}^{(\mathrm{P3})} = \ell_{\mathrm{h}}(a)A - \ell_{\mathrm{h}}(a)B \tag{45a}$$

$$s_{\mathrm{h},\sigma(2,3),a}^{(\mathrm{P3})} = s_{\mathrm{h},\mathrm{uu},a}^{(\mathrm{P3})} = -\ell_{\mathrm{u}}(a)A + \ell_{\mathrm{u}}(a)B \tag{45b}$$

$$s_{\mathrm{h},\sigma(3,3),a}^{(\mathrm{P3})} = s_{\mathrm{h},\mathrm{ud},a}^{(\mathrm{P3})} = -\ell_{\mathrm{u}}(a)A \tag{45c}$$

$$s_{\mathrm{h},\sigma(4,3),a}^{(\mathrm{P3})} = s_{\mathrm{h},\mathrm{hd},a}^{(\mathrm{P3})} = -\ell_{\mathrm{h}}(a)B. \tag{45d}$$

Hence, for instance, dividing Eqs. (43a) and (43b), or (43d) and (43c), we obtain the sensitivity ratios

$$\frac{s_{\mathrm{h},\mathrm{hh},a}^{(\mathrm{P1})}}{s_{\mathrm{h},\mathrm{uh},a}^{(\mathrm{P1})}} = \frac{s_{\mathrm{h},\mathrm{hu},a}^{(\mathrm{P1})}}{s_{\mathrm{h},\mathrm{uu},a}^{(\mathrm{P1})}} = \frac{\ell_{\mathrm{h}}(a)}{\ell_{\mathrm{u}}(a)}. \tag{46a}$$

Also, from Eqs. (43a) and (44a), or from Eqs. (43c) and (44c), we have a sensitivity relationship between P1 and P2

$$s_{\mathrm{h},\mathrm{hh},a}^{(\mathrm{P1})} + s_{\mathrm{h},\mathrm{hd},a}^{(\mathrm{P2})} = s_{\mathrm{h},\mathrm{uu},a}^{(\mathrm{P1})} + s_{\mathrm{h},\mathrm{ud},a}^{(\mathrm{P2})} = 0. \tag{46b}$$

Similarly, from Eqs. (43b) and (45c), or from Eqs. (43d) and (45d), we have a relationship between the sensitivities of P1 and P3

$$s_{\mathrm{h},\mathrm{uh},a}^{(\mathrm{P1})} + s_{\mathrm{h},\mathrm{ud},a}^{(\mathrm{P3})} = s_{\mathrm{h},\mathrm{hu},a}^{(\mathrm{P1})} + s_{\mathrm{h},\mathrm{hd},a}^{(\mathrm{P3})} = 0. \tag{46c}$$

Along the same lines, from Eqs. (44b) and (45b), or from Eqs. (44c) and (45a), we have a relationship between P2 and P3

$$s_{\mathrm{h},\mathrm{uh},a}^{(\mathrm{P2})} + s_{\mathrm{h},\mathrm{uu},a}^{(\mathrm{P3})} = s_{\mathrm{h},\mathrm{hu},a}^{(\mathrm{P2})} + s_{\mathrm{h},\mathrm{hh},a}^{(\mathrm{P3})} = 0. \tag{46d}$$

From Eqs. (44b), (43b), and (43c), we can also write a sensitivity under P2 in terms of sensitivities under P1

$$s_{\mathrm{h},\mathrm{uh},a}^{(\mathrm{P2})} = s_{\mathrm{h},\mathrm{uh},a}^{(\mathrm{P1})} - s_{\mathrm{h},\mathrm{uu},a}^{(\mathrm{P1})}. \tag{46e}$$

From Eqs. (44d), (43a), and (43d), we can similarly write another sensitivity under P2 in terms of sensitivities under P1

$$s_{\mathrm{h},\mathrm{hu},a}^{(\mathrm{P2})} = -s_{\mathrm{h},\mathrm{hh},a}^{(\mathrm{P1})} + s_{\mathrm{h},\mathrm{hu},a}^{(\mathrm{P1})}. \tag{46f}$$

From Eqs. (45a), (43a), and (43d), we can further write a sensitivity under P3 in terms of sensitivities under P1

$$s_{\mathrm{h},\mathrm{hh},a}^{(\mathrm{P3})} = s_{\mathrm{h},\mathrm{hh},a}^{(\mathrm{P1})} - s_{\mathrm{h},\mathrm{hu},a}^{(\mathrm{P1})}, \tag{46g}$$

and from Eqs. (45b), (43b), and (43c), we can further write another sensitivity under P3 in terms of sensitivities under P1

$$s^{(\mathrm{P3})}_{\mathrm{h},uu,a} = -s^{(\mathrm{P1})}_{\mathrm{h},uh,a} + s^{(\mathrm{P1})}_{\mathrm{h},uu,a}. \tag{46h}$$

Aside from giving us a healthy dose of ambiguity regarding the drivers of HLE differences, these symmetries can be used to help translate the sensitivities of one parameterization to those of another without the need for further re-framing or recalculation of the whole sensitivity exercise. Especially for HLE calculated using P1 (the transitions widely used in the matrix algebra approach), one might nonetheless wish to discuss decomposition results in terms of P2 sensitivities.

Data

Lievre et al. [2003] give parameters for their fitted model of transition probabilities in their Table 2. We convert these parameters directly to monthly transition probabilities for ages 50 through 110 for males and females using their equation 33 and the ALR inverse implementation of the {compositions} R package [van den Boogaart et al., 2022]. We then convert these to annual transition probabilities by raising age-step submatrices to the 12th matrix power using the {expm} R package [Goulet et al., 2021]. We display the resulting transition probabilities in Fig. 2.

Results

Using the transitions estimated by Lievre et al. [2003] we calculate health expectancies. We compute the initial conditions at age 50 ($\ell_i(0)$, where 0 here means age 50) assuming that the probabilities we have for age 50 are constant in younger ages. With these initial conditions, the probabilities of Fig. 2 imply the expectancies shown in Table 1, where HLE is specifically disability free life expectancy (DFLE) and ULE is disabled life expectancy (DLE).

Table 1: Expectancies derived from Lievre et. al. (2003)

Sex	DFLE	DLE	LE
Females	27.15	2.84	29.99
Males	25.69	1.67	27.37
Difference	1.46	1.17	2.62

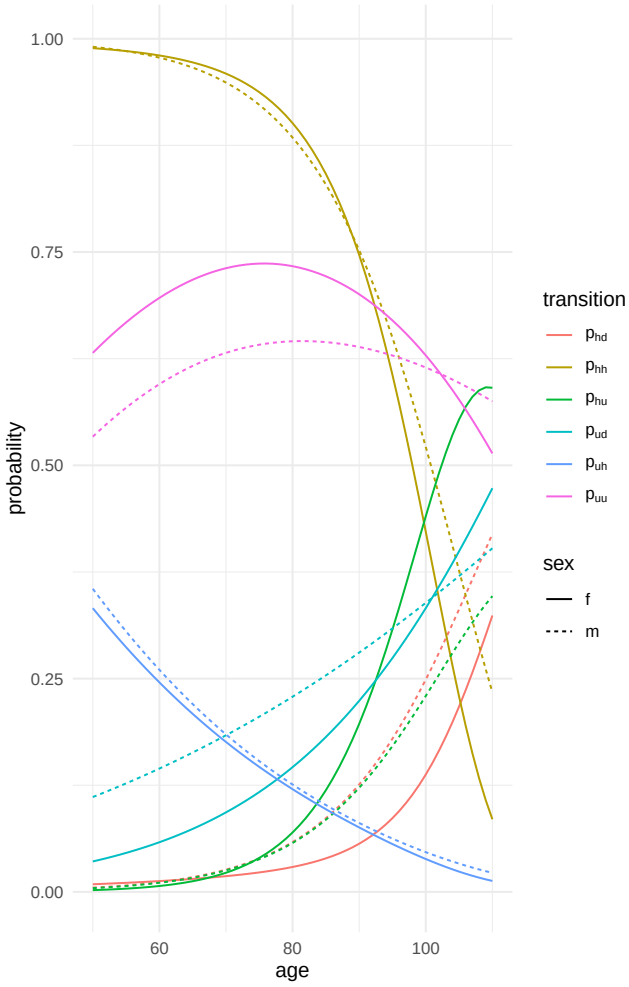


Figure 2: Transition probabilities re-computed from Lievre et al. [2003]

Sex differences shown in the last row of Table 1 are the differences we now aim to decompose in terms of Lievre et al.'s [2003] transition probabilities. Differences in expectancies are due to transition parameter differences (see appendix Fig. 5), but their net contribution to the difference depends on the model parameterization.

Applying the Eq. (22) for the three parameterizations (P1 = "no-death", P2 = "no-self", P3 = "no-health"), we generate the sensitivities displayed in Fig. 3 (only females shown), where one may visually verify that sensitivities for the same transition vary in important ways depending on which parameter case is used. Some of the symmetries

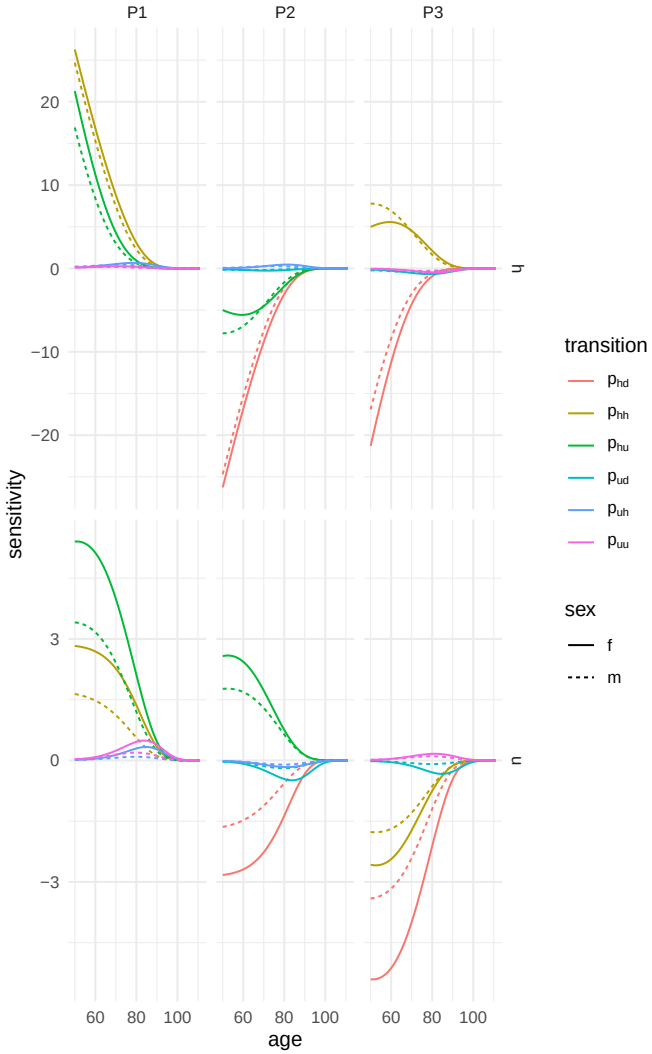


Figure 3: Sensitivities for women, for parameterizations P1, P2, and P3, for disability free life expectancy (DFLE) and disabled life expectancy (DLE), re-computed from Lievre et al. [2003]. Note the different scales in the vertical axes for DFLE and DLE.

and equalities mentioned in Eq. (46) may also be spotted in Fig. 3. We show the sensitivities for DFLE and DLE (note vertical axis different ranges), and omit LE, which is also the sum of the other two and looks qualitatively like that of DFLE in this example. Male and female sensitivities for all expectancies can be found in Appendix B.

To explain why it is that males have 1.46 fewer years lived disability-free than females, we should multiply a sensitivity by the corresponding difference in transitions per Eq. (26). Clearly, from the results in Fig. 3, we know that main findings will be heavily dependent on the parameterization used to calculate the sensitivity (P1-P3). Following Eq.(26) the decomposition results as shown in Fig. 4 (here only shown for female DFLE) are indeed strikingly different.

Not only do age patterns of decomposition results differ depending on which parameterization we choose, but marginal sums for the given transition types differ, per Table 2.

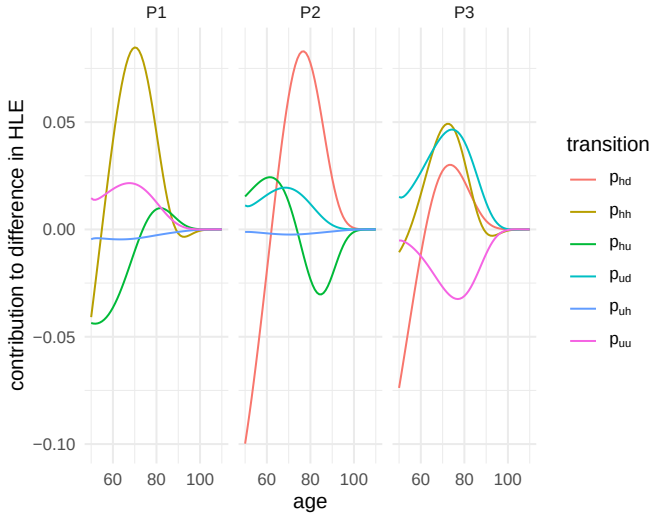


Figure 4: Decompositions of the sex difference (female - male) for parameterizations P1, P2, and P3, for disability free life expectancy (DFLE) and disabled life expectancy (DLE), re-computed from Lievre et al. [2003].

Transition	P1	P2	P3
p_{hd}		0.95	0.07
p_{hh}	1.53		0.89
p_{hu}	-0.57	0.01	
p_{ud}		0.55	1.32
p_{uh}	-0.15	-0.08	
p_{uu}	0.62		-0.85
Init. Cond.	0.02	0.02	0.02
Total	1.45	1.45	1.45
Residual	0.00	0.00	0.00

Table 2: Marginal sums of the transition components of the decomposition of the sex gap in DFLE based on transitions derived from Lievre et al. [2003].

The total effects of different transitions in Table 2 lead to qualitatively different understandings and, therefore, might recommend different intervention priorities. Note that for each parameterization we arrive at the same effect of the initial state mixture (more favorable for females), and the same total over all transitions. The remaining residual is comparatively small (≈ 0.001).

Discussion

A case for P2 decomposition of HLE

In each of the three parameterizations considered, one type of transition is lacking: For P1 this is mortality, P2 lacks self-transitions, and P3 lacks health-transitions. Refer to the DFLE (HLE) sensitivities depicted in the top panel of Fig. 3, specifically the green lines, which refer to the sensitivity to a perturbation in p_{hu}, a transition only included in parameterizations 1 and 2. For P1, this sensitivity is positive, but for P2, it is negative. That is, under P1 an increase in p_{hu} increases DFLE, whereas under P2, it decreases DFLE. What are we to make of this? Each

is mathematically correct, given the respective recursion systems P1 and P2, but only one of these two stands up to substantive scrutiny: the effect of increasing disability onset is unambiguously to decrease DFLE, not increase it. Consider an axiom: exiting a state cannot lead to increased time spent in the state being left. For our data application, women had a lower disability onset before age 74 and a higher onset in older ages; hence, the decomposition effect of this transition seen in Fig. 4 only gives the expected pattern for P2: a positive contribution to the female DFLE advantage before the crossover age, and a negative contribution thereafter. In the net, for P2, the small female advantage in younger ages has a slightly larger positive effect than the disadvantage in older ages. The unintuitive opposite would hold for P1. A similar argument could be made for the recovery transition p_{uh}, which is harder to see in the figures as it plays a smaller role. On the other hand, each transition in P2 has an intuitive sensitivity sign: increases in mortality reduce occupancy times, and increases in health transitions out of origin state i reduce occupancy time in i and increase it in destination state j. The sensitivity of HLE to increases in p_{uu} and the sensitivity of ULE to increases in p_{hh} are each inconsistent between P1 and P3. An increased probability of staying in state i has an ambiguous effect on occupancy time in the other state j: remaining in state i might (1) increase the chance of future transitions to j, thereby indirectly increasing time in j, or it might (2) increase the chances of death in i, at the expense of potential time spent in j.

A softer and simpler argument for P2 may also be made, at least for models of health and mortality: If one finds P3 unsettling, due to its lack of health transitions — after all, this is a model of health and mortality — then P1 may now strike the same feeling for its lack of mortality, at least for this family of research questions. Specifically, for purposes of decomposition, treating attrition as the fundamental force lends itself to straightforward interpretation for the present application of multistate models. Other sorts of substantive arguments for preferring different parameterizations are also possible, and by no means do we rule them out, especially but not only for other domains of multistate model application.

A decomposition based on P2 will always attribute differences to mortality or health transitions. This is also the closest analog among the three parameterizations to what a direct decomposition of Sullivan-style [Sullivan, 1971] inputs attempts to achieve: differences are due to mortality (a life table envelope) or health (prevalence) [Andreev et al., 2002, Nusselder and Looman, 2004, Shkolnikov et al., 2017], strong endogeneity between the mortality and prevalence notwithstanding. To the extent that transitions are well-estimated, the decomposition of incidence-based HLE using P2 gives more reliable and actionable information than a Sullivan-style decomposition: Mortality components are separated by states, and health is represented by flows (transitions) rather than a stock variable (prevalence).

Comparison with other approaches

There are two versions of multistate decomposition of HLE that we know about [Shen et al., 2023, Moretti et al., 2023].

Shen et al. [2023] also decomposed a sex gap in HLE using data from Payne [2022], following an approach analogous to P1. According to our symmetry findings (compare e.g. Eqs. (43a) and (44a)), their interpretation of the effects of remaining healthy or unhealthy was correctly interpreted as due to mortality differences, although the respective magnitudes will be somewhat different due to differently composed $\boldsymbol{\delta}$ used in Eq. (26), which destroys the symmetry. Referring to our own example, this loss of symmetry is somewhat visible in comparing the lines for e.g. P1 p_{hh} and P2 p_{hd} in Fig. 4, but it is easier to spot in Table 2 where we see the marginal total for p_{hh} in P1 is 1.51, whereas the corresponding p_{hd} effect in P2 amounts to 0.95, despite these two transitions having exactly symmetrical sensitivities between P1 and P2.

More problematically, the overall effects of onset and recovery (p_{hu} and p_{uh}, respectively) are very different from those of P2. In the Shen et al. [2023] data, females show a higher probability of disability onset (p_{hu}) in all ages, yet the effect was interpreted to have contributed to the sex-gap in HLE rather than attenuate it, as per P2 and our own intuition. As described in the section on symmetries, one could translate the Shen et al. [2023] results to an attrition-only interpretation (P2) with a few simple steps. We offer a short reanalysis of the Shen et. al. results in the code repository.

Moretti et al. [2023] take a decomposition approach based on P2, but using the Horiuchi et al. [2008] method, which eliminates the need for an analytic sensitivity, albeit at high computational cost, especially if confidence intervals are desired. Our analytic approach is more computationally efficient if bootstrapping is required. Point estimates of decomposition effects should be consistent between the two approaches.

General guidance

As mentioned, each of the three parameterizations results in decompositions with consistent overall sums, but most likely slightly off from the observed difference in expectancies (too small for two decimal places in Table 2). This residual can be forced to 0 either (i) by calculating the sensitivity from parameters at a well-selected intermediate point (we used the average of females and male health and mortality transitions, see Caswell [2001], p. 261-262) or (ii) by repeating the exercise interpolated over small intervals, partly borrowing the linear integral strategy from Horiuchi et al. [2008], in which case the residual approaches 0 as the number of interpolation steps between parameter sets increases. The researcher should weigh whether such steps are really necessary, as these will add both programming and computational overhead to the problem, whereas the size of the residual is usually trivial. We give `R` code demonstrating how to implement both approaches in the reproducibility repository, and we find that the optimization strategy (i) is more computationally

efficient than the interpolation strategy (ii).

Marginal age patterns of decomposition results are equal between the different parameterizations, and we give `R` code to demonstrate this in the repository. Knowing the overall age pattern explaining an observed difference in health expectancy is not particularly informative beyond contextualizing more detailed results. Unlike an all-cause life expectancy decomposition, where it is sufficient to determine the effects of a particular age group, for multistate processes, we would like to know the influence of each transition.

One should bear in mind, especially if these methods are used to decompose overall life expectancy (LE), that results are expected to vary depending on which health conditions are considered. In our example (See appendix Table 4), we see that sex differences in LE are almost entirely explained by mortality, whereas contributions from health transitions balance out.

In this paper, we only treat the case of two transient and reversible states, a rather simple model as multistate models go. Further work is required to generalize our analytic solution to a more general representation of state spaces. Our recommendation to use the attrition-only parameterization (P2) for sensitivities and decomposition applies only to models of health and mortality. Other applications of multistate models should carefully consider which parameterization produces the soundest decomposition results.

Conclusions

We describe a problem in the decomposition of discrete time multistate models that to our knowledge, has never been described in the literature: Decomposition results and interpretations depend on which transition probabilities are used to calculate the expectancy (or other synthetic index). We have treated this problem for one of the simplest recurrent multistate models: a health model with two reversible health states and death. We give three redundant parameterizations standing for different subsets of transition parameters to calculate the same value of health expectancy. For each of these parameterizations, we give corresponding formulas to calculate parameter sensitivities for expectancies. We show that the sensitivity of HLE (or ULE or LE) to a given transition can be very different, depending on the parameterization used to calculate it, and that this inequality between parameterizations translates to inequality between decomposition results. Therefore, researchers must take care when interpreting decomposition results from multistate models. Certain symmetries and equalities exist between the sensitivities under different parameterizations, which can be exploited to translate sensitivities (and, by extension, decomposition results) under one parameterization to those under another. We argue that the interpretation of HLE and related decomposition results is best for attrition parameters (P2). We use transition probabilities recovered from Lievre et al. [2003] to illustrate these concepts.

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CRediT author statement

Tim Riffe Conceptualization, Investigation, Funding acquisition, Methodology, Software, Validation, Formal analysis, Writing - Original draft preparation, reviewing & editing. **Francisco Villavicencio** Validation, Writing - reviewing & editing. **Mauricio Gonzalez** Methodology, Validation, Formal analysis, Writing - Original draft preparation, reviewing & editing.

Competing interest

The authors declare that they have no competing interests.

Data and materials availability

All data and code needed to evaluate the conclusions in the paper are given in the following anonymized OSF repository `https://osf.io/uwzt2/?view_only=e3c57efb3d214b8db013d3b74b5fb3a0`. This link will change after paper acceptance.

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Appendices

A Parameter differences

In this appendix, we show sex differences in transition parameters, the $\boldsymbol{\delta}$ from e.g. Eq. 26.

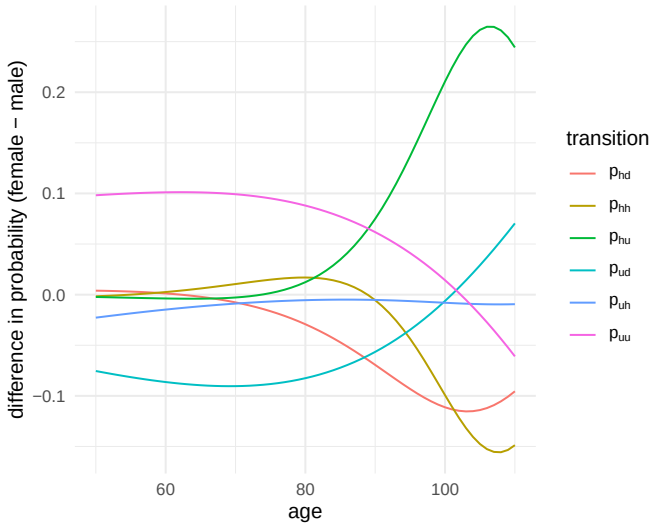

Figure 5: Differences in transition probabilities (female - male) re-computed from Lievre et al. [2003]

B All sensitivities

In this appendix, we show all sensitivities for DFLE, DLE, LE, and all three parameterizations for both males and females. The values depicted in Fig. 6 are also available in the main repository in the file `all_sensitivities.csv`. Sensitivity to initial conditions are not depicted in the Figure. These values are given in Table 3.

	sex	DFLE	DLE	LE
1	f	4.87	-2.56	2.31
2	m	7.80	-1.77	6.03

Table 3: Sensitivities to initial conditions for parameterizations P1, P2, and P3, for disability free life expectancy (DFLE) and disabled life expectancy (DLE), and total life expectancy (LE) for males and females re-computed from Lievre et al. [2003]

C All decompositions

In this appendix, we show sex-decompositions for all three parameterizations of DFLE, DLE, LE. The values depicted in Fig. 7 are also available in the main repository in the file `all_decompositions.csv`. Effects due to initial conditions are identical for the three parameterizations.

This next table is generated excluding the computed effects of initial conditions. Notice the far smaller residuals. Possibly need to rethink initial conditions sensitivity calculations.

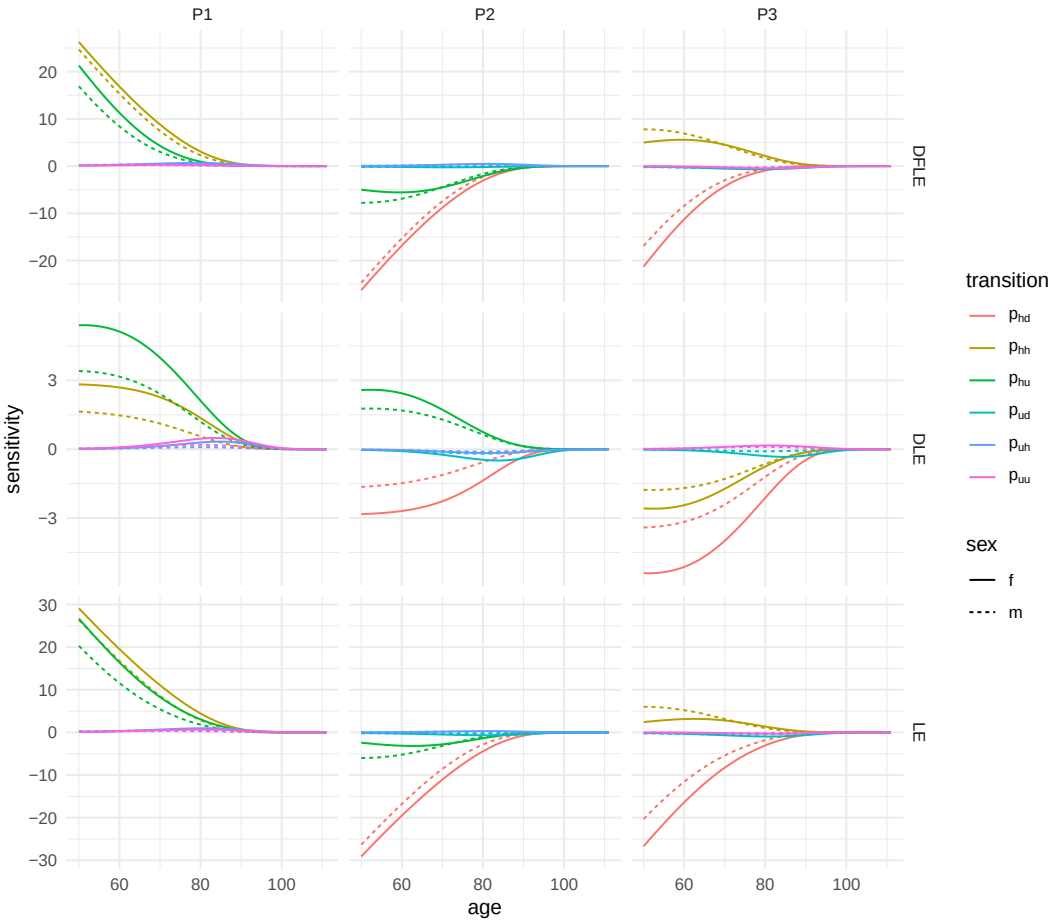

Figure 6: Sensitivities for parameterizations P1, P2, and P3, for disability free life expectancy (DFLE), disabled life expectancy (DLE), and total life expectancy (LE) for males and females using parameters re-computed from Lievre et al. [2003] Vertical axis ranges vary by expectancy.

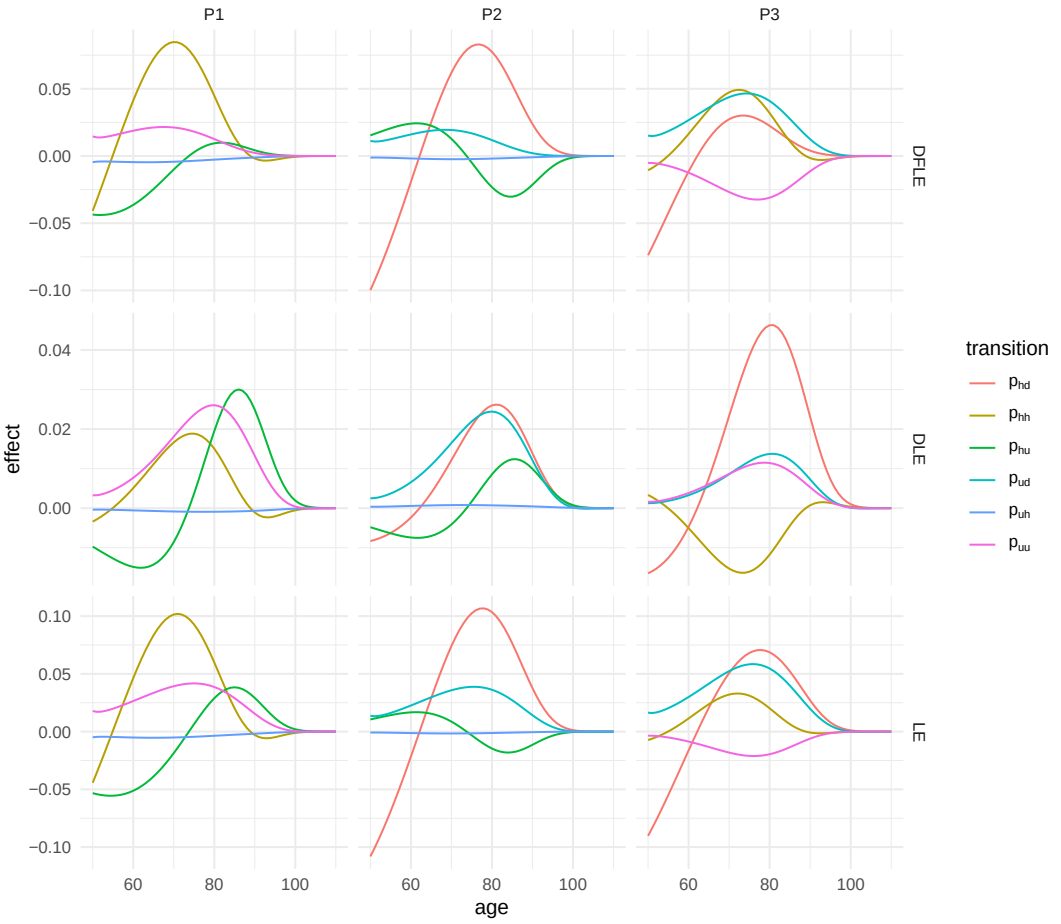

Figure 7: Sex decompositions for parameterizations P1, P2, and P3, for disability free life expectancy (DFLE), disabled life expectancy (DLE), and total life expectancy (LE), using parameters calculated from Lievre et al. [2003] Vertical axis ranges vary by expectancy.

Expectancy	Component	P1	P2	P3
DFLE	p_{hd}		0.95	0.07
	p_{hh}	1.53		0.89
	p_{hu}	-0.57	0.01	
	p_{ud}		0.55	1.32
	p_{uh}	-0.15	-0.08	
	p_{uu}	0.62		-0.85
	Init. Cond.	0.02	0.02	0.02
	Total	1.45	1.45	1.45
	Residual	0.00	0.00	0.00
DLE	p_{hd}		0.46	0.80
	p_{hh}	0.34		-0.30
	p_{hu}	0.17	0.05	
	p_{ud}		0.61	0.34
	p_{uh}	-0.03	0.03	
	p_{uu}	0.67		0.30
	Init. Cond.	-0.01	-0.01	-0.01
	Total	1.14	1.14	1.14
	Residual	0.03	0.03	0.03
LE	p_{hd}		1.41	0.87
	p_{hh}	1.86		0.59
	p_{hu}	-0.40	0.05	
	p_{ud}		1.16	1.66
	p_{uh}	-0.18	-0.05	
	p_{uu}	1.30		-0.55
	Init. Cond.	0.02	0.02	0.02
	Total	2.59	2.59	2.59
	Residual	0.03	0.03	0.03

Table 4: Transition margins from sex decompositions for parameterizations P1, P2, and P3, for disability free life expectancy (DFLE), disabled life expectancy (DLE), and total life expectancy (LE), using parameters calculated from Lievre et al. [2003]