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Short communication

On statistical and information-based virtual age of degrading systems

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Abstract

Two approaches to defining a virtual age of a degrading system are considered. The first one is based on the fact that deterioration depends on the environment. In a more severe environment deterioration is more intensive, which means that objects are aging faster and therefore, the corresponding virtual age is larger than the calendar age in a baseline environment. The second approach is based on considering an observed level of individual degradation and comparing it with some average, 'population degradation'. The virtual age of the series system is defined. Several illustrative examples are considered.

Keywords: Virtual age; Degradation; Aging distributions; Failure rate; Mean remaining lifetime; Repairable systems

1. Introduction

Lifetimes of degrading (deteriorating) systems can be effectively modeled by aging distributions. The simplest and probably the most natural is the class of distributions with increasing failure rates (IFR). For instance, various mechanical devices with wear accumulation follow the Weibull law with increasing failure rate. It is also well known that adult aging in humans is usually modeled with the help of the Gompertz law (exponentially increasing failure rate). In this paper we implicitly assume the IFR property, although some generalizations can be also considered.

It is clear that an age of a system as some overall trivial marker of deterioration, is really informative in some sense only for degrading objects. This age is the same for all individuals in a population, which are simultaneously incepted into operation. We shall call this chronological age the statistical age.

Deterioration usually depends on environment. Deterioration under a more severe environment is more intensive, which means that objects are aging faster. Therefore, in Section 3 we discuss a statistical virtual age, which is defined for degradation comparison in different regimes (stresses) via the corresponding scale transformation. Degradation, however is a stochastic process, therefore, individuals age differently. Assume for simplicity that deterioration of a system can be modeled by a single predictable increasing stochastic process. Observation of a state of a system at time t can give an indication (under certain assumptions) of its age defined by the level of deterioration. We shall call this characteristic an information-based virtual age of a system (whereas, in fact, this is a 'real individual age' in some sense). If for, instance, a person of 50 years old looks like and has vital characteristics (blood pressure, level of cholesterol, etc.) as of a 30 years old one, we can say that this observation indicates that his virtual age is 30. In Section 2 we discuss the Kijima [1] model for the virtual age in repairable systems. In Sections 3 and 4 under rather stringent assumptions we consider other possible heuristic approaches to defining the virtual age of a system.

Another challenging problem is to define the virtual age of a system with components in series having different virtual ages. In a conventional setting all components are of the same chronological age and therefore this problem does not exist. We discuss possible ways of weighting virtual ages for a series system in Section 5.

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2. Virtual age in repairable systems

We start with some introductory considerations on a notion of a virtual age for repairable systems, which was introduced by Kijima [1] (see also [2,3]). Although the next sections deal with systems without repair, the forthcoming reasoning can be helpful for the rest of this paper.

A convenient mathematical description of repair processes uses a concept of stochastic (or failure) intensity [4]. Consider, for example, a renewal process (perfect instantaneous repair) with underlying distribution $F(t)$ and the failure rate $\lambda(t)$:

$$\lambda_t = \sum_{n=0}^{\infty} \lambda(t - T_n) I(T_n \leqslant t < T_{n+1}), \tag{1}$$

where the indicator, as usually, is defined as: $I(t \in B) = 1$ and $I(t \notin B) = 0$, $B = [a, b) \subset [0, \infty)$. Denote by A_t the age process, which corresponds to the renewal process (1):

$$A_t = \sum_{n=0}^{\infty} (t - T_n) I(T_n \leqslant t < T_{n+1}). \tag{2}$$

Thus, this stochastic process starts at $t = 0$ as a linear function with a unit slope. It jumps again to 0 at T_1 the time of the first repair, etc. The age of a repairable system in this case is just time elapsed since the last repair.

As a minimal repair does not change the age of a system, the corresponding age process is deterministic:

$$A_t = t.$$

Considering intermediate between the perfect and minimal level of repair brings an important notion of a virtual age. Assume that repair at $t = t_1$ (realization of T_1) decreases the age of a system not to 0 as in the case of a perfect repair, but to $v_1 = qt_1$, $0 < q < 1$, and the system starts the second cycle with this initial age in accordance with the Cdf $1 - \overline{F}(t_1 + t)/\overline{F}(t_1)$, where $\overline{F} \equiv 1 - F$. This age is called the virtual age. The forthcoming cycles are defined in a similar way to form a process of general repair [1,3]. It is clear that the approach is reasonable, when the repair action decreases the level of degradation. Therefore, the failure rate $\lambda(t)$ is assumed to be an increasing function and the virtual age in this approach uniquely defines the level of degradation after the repair.

3. Statistical virtual age

Consider a degrading system in a fixed baseline environment with the Cdf of time to failure $F_b(t)$. As it was mentioned, the chronological age t will be called the statistical age. Let the second statistically identical system be operating, e.g., in a more severe environment (regime) with the Cdf of time to failure $F_s(t)$. We want to establish an age correspondence between the systems in two regimes considering the baseline as a reference one. It is clear that degradation under the second regime is more intensive therefore, the time for a system under the baseline regime to reach the same level of degradation, as the system, that had operated for the same time t under the second regime, will be larger than t. We shall call this time the statistical virtual age of the second system. Note, that this definition can be applied only when we have different regimes and the baseline environment in this case is considered as a reference one. Now we will describe this definition in mathematical terms for a specific but a very important in applications model.

Assume that the lifetimes for two regimes are ordered in the sense of a usual stochastic ordering [5]:

$$\overline{F}_s(t) < \overline{F}_b(t), \quad t \in (0, \infty), \tag{3}$$

where, as usual, $\overline{F}(t) \equiv 1 - F(t)$. This assumption is reasonable, as the second regime is more severe than the baseline one.

Inequality (3) implies the following equation:

$$F_s(t) = F_b(W(t)), \quad W(0) = 0, \quad W(t) > t, \quad t \in (0, \infty). \tag{4}$$

Relation (4) can be interpreted as a general Accelerated Life Model (ALM) [6,7] with a scale transformation function $W(t)$, which in its turn can be interpreted as an additive degradation function (in fact, this is another assumption for our degrading systems):

$$W(t) = \int_0^t w(u)\,\mathrm{d}u$$

where $w(t)$ has a meaning of a speed of degradation (for convenience of comparison we assume that in the baseline environment this speed is equal to 1).

Given our notions of the statistical age and the statistical virtual age, model (4) suggests that the statistical virtual age of a system, which was operating for time t in a more severe environment, is $W(t)$ compared with the statistical age t of a system in a baseline environment. Formally:

Definition 1. The function $W(t)$ in the ALM model describing the relative degradation for two regimes is called the statistical virtual age for this model.

It follows from (4) that the failure rate of a system under the second regime is

$$\lambda_s(t) = w(t)\lambda_b(W(t)). \tag{5}$$

If $\lambda_b(t)$ is increasing, then the crude sufficient condition for $\lambda_s(t)$ to increase is that $w(t)$ should not decrease.

When the failure rates (or the corresponding Cdfs) are given, or estimated from the data, relation (4) can be viewed as an equation for obtaining the statistical virtual age $W(t)$:

$$\exp\left\{-\int_0^t \lambda_s(u)\,\mathrm{d}u\right\} = \exp\left\{-\int_0^{W(t)} \lambda_b(u)\,\mathrm{d}u\right\}$$
$$\Rightarrow \int_0^t \lambda_s(u)\,\mathrm{d}u = \int_0^{W(t)} \lambda_b(u)\,\mathrm{d}u. \tag{6}$$

Hence $W(t)$ is uniquely defined by Eq. (6) as a monotonically increasing function of t: $W(t) \to \infty$, while $t \to \infty$; $W(0) = 0$.

Example 1. Let the failure rates in both regimes be increasing power functions

$$\lambda_b(t) = \alpha t^{\beta}, \quad \lambda_s(t) = \mu t^{\eta}, \quad \alpha, \beta, \mu, \eta > 0.$$

The Weibull distribution is often used for lifetime modeling of degrading objects. Then the corresponding statistical virtual age is defined by Eq. (6):

$$W(t) = \left(\frac{\mu(\beta+1)}{\alpha(\eta+1)}\right)^{1/(\beta+1)} t^{(\eta+1)/(\beta+1)}.$$

In order inequality $W(t) > t$ for $t > 0$ to hold, the following restrictions on parameters are sufficient: $\eta \geqslant \beta$; $\mu(\beta+1) > \alpha(\eta+1)$.

4. Information-based virtual age

In the previous section no additional information on a system's deterioration was available, therefore a system was considered as a black box. However, deterioration is a stochastic process, therefore individuals age differently. Observation of a state of a system at time t can give an indication (under certain assumptions) of its age defined by the level of deterioration.

We start with a meaningful reliability example, which will help to understand the issue of the information-based virtual age. The number of operable components at the time of observation defines the corresponding level of deterioration in this example.

Example 2. Consider a system of $n+1$ components (one main component and n cold standby identical ones) with constant failure rates λ. Denote the system's lifetime random variable by T_{n+1}. The corresponding Cdf is

$$F_{n+1}(t) \equiv Pr[T_{n+1} \leqslant t] = 1 - \exp\{-\lambda t\} \sum_0^n \frac{(\lambda t)^i}{i!} \tag{7}$$

with an increasing failure rate [8]:

$$\lambda_{n+1}(t) = \frac{\lambda \exp\{-\lambda t\}(\lambda t)^n / n|}{\exp\{-\lambda t\}\sum_0^n (\lambda t)^i / i!}. \tag{8}$$

Note [9], that the failure rates for different numbers of standby components are ordered as

$$\lambda_{k_1}(t) > \lambda_{k_2}(t). \tag{9}$$

for $0 < k_1 < k_2 \leqslant n+1$; $\forall t \in (0, \infty)$.

The number of failed components observed at time t is a natural measure of accumulated degradation in $[0, t]$ for this setting. In order to obtain the corresponding information-based virtual age to be compared with the statistical age t, consider, firstly, the following conditional expectation:

$$D(t) \equiv E[N(t)|N(t) \leqslant n] = E[N(t)|T_{n+!} > t] = \frac{\exp\{-\lambda t\}\sum_0^n i((\lambda t)^i / i!)}{\exp\{-\lambda t\}\sum_0^n (\lambda t)^i / i!}, \tag{10}$$

where $N(t)$ is the number of events in the interval $[0, t]$ in the Poisson process with rate λ. Relation (10) defines the expected value of the number of failures (measure of degradation) for the population of systems that had survived in $[0, t]$. The function $D(t)$ is monotonically increasing, $D(0) = 0$ and $\lim_{t\to\infty} D(t) = n$, which follows from Eq. (10) and a simple general reasoning as well. It is worth mentioning that $E[N(t)] = \lambda t$, which exhibits the different from $D(t)$ shape. This function defines an average degradation curve for the system. If our observation $0 \leqslant k \leqslant n$ the number of failed components at time t lies on this curve, then the information-based virtual age is just equal to the statistical age t.

Denote the information-based virtual age by $V(t)$. Our definition for this specific model is:

$$V(t) = D^{-1}(k). \tag{11}$$

If $k = D(t)$, then: $V(t) = D^{-1}(D(t)) = t$. Similar:

$$k < D(t) \Rightarrow V(t) \langle t, \quad k \rangle D(t) \Rightarrow V(t) > t$$

Thus, in this example, the information-based virtual age can be obtained in a simple way.

Note that the approach of this example can be generalized in principle to the case of IFR components. As operation of convolution preserves the IFR property, $\lambda_n(t)$, $n = 1, 2, \ldots$ is also increasing in this case and therefore the system is degrading even in this sense. Relation becomes

$$D(t) \equiv E[N(t)|N(t) \le n] = E[N(t)|T_{n+!} > t] = \frac{\sum_0^n i(F^i(t) - F^{i+1}(t))}{1 - F^n(t)},$$

where $F^i(t)$ denotes the i-fold convolution of $F(t)$ with itself [5]. The function $D(t)$ in this case has the same meaning as in the specific exponential one. It increases and tends to n as $t \to \infty$.

The general case of degrading objects can be considered in the same line. Let D_t be an increasing, smoothly varying (predictable) stochastic process of degradation with a mean $D(t)$, which defines the statistical age of our object as t. We also assume for simplicity that this is a process with independent increments, and therefore it possesses the Markovian property. Similar to (11), observation, d_t at time t defines the information-based virtual age. Formally:

Definition 2. Let D_t be an increasing, with independent increments, smoothly varying stochastic process of degradation with a mean $D(t)$, and d_t be an observation at time t. Then the information-based virtual age is

$$V(t) = D^{-1}(d_t).$$

If the failure time T is the stopping time for the degradation process D_t, then, similar to Eq. (10), the conditional expected degradation $E[D_t|T>t]$ should be used as the function $D(t)$.

Alternative, and somehow more justified way of introducing $V(t)$ is via the information-based remaining lifetime [9]. The statistical (conventional) mean remaining lifetime (MRL) at t of a system with the Cdf $F(x)$ is defined in a standard way as:

$$M(t) = \int_0^\infty F(x|t)\,\mathrm{d}x \equiv \int_0^\infty \frac{\overline{F}(t+x)}{\overline{F}(t)}\,\mathrm{d}x \tag{12}$$

and we can compare it with the mean information-based remaining lifetime, denoted by $M_I(t)$. Given d_t the observed level of degradation at time t as a new initial value for a degradation process, $M_I(t)$ defines the mean time to failure in this case. Therefore, this characteristic is similar to $M(0)$(and not to $M(t)$, $t>0$), but with a different initial value of degradation. In Example 2: $d_t = k$ the number of the failed components, which means that $M_I(t) = (n+1-k)/\lambda$.

Definition 3. The information-based virtual age of a degrading system is given by the following equation:

$$V(t) = t + (M(t) - M_I(t)). \tag{13}$$

Thus, the virtual age is the chronological (statistical) one plus the difference between the statistical and the information-based mean remaining lifetimes. If, e.g., $M(t) = t_1 < t_2 = M_I(t)$, then $V(t) = t - (t_2 - t_1) < t$: and we have additional expected $t_2 - t_1$ years of life for an object, as compared with the 'no information' version. It is easy to show that under natural assumptions $M_I(t) - M(t) < t$, which ensures that $V(t)$ is positive. Indeed, as $\mathrm{d}M(t)/\mathrm{d}t > -1$ (see, e.g., [10]), the function $t + M(t)$ is increasing. Therefore:

$$t + M(t) > M(0).$$

Thus, it is sufficient to assume that $M_I(t) \leqslant M(0)$ (a kind of information-based NBUE property, see e.g., [11]) which holds, for instance, for increasing processes of deterioration with independent increments and the threshold, as in Example 2, where $M_I(t) = (n+1-k)/\lambda$ and $M(0) = (n+1)/\lambda$.

Note that the idea of our definition (13) is in adding (subtracting) to the chronological age t the gain (loss) in the remaining lifetime due to additional information on the state of a degradation process at time t.

Example 3. Consider a system of 2 i.i.d components in parallel with exponential Cdfs. Then: $\overline{F}(t) = \exp\{-2\lambda t\} - 2\exp\{-\lambda t\}$ and

$$M(t) = \int_0^\infty \frac{2\exp\{-\lambda t\} - \exp\{-2\lambda t\}\exp\{-\lambda x\}}{2 - \exp\{-\lambda x\}}\,\mathrm{d}x.$$

Assume that our observation at time t is: two operable components. Then $M_I(t) = \int_0^\infty \overline{F}(x)\,\mathrm{d}x = 1.5\lambda$ and in accordance with (13) (and taking into account that $M(t)$ is a decreasing function): $0 < V(t) < t$, which means that the information-based virtual age in this case is smaller than the statistical age t. If observation is one operable component, then the virtual age is larger than t: $V(t) > t$. Note, that $M_I(t)$ does not depend on t in this example due to the memoryless property of exponential distribution.

5. Virtual age in a series system

In this section possible approaches to defining a virtual age of a series system of degrading components with different virtual ages will be considered. In a conventional setting all components have the same chronological age and therefore this problem does not exist. However, it is really important in different applications (specifically, biological) to obtain a virtual age of a series system. For instance, assume that there are two components in series and the first one has a much higher relative level of degradation that the first one, meaning that the corresponding virtual ages are also different. How to define the virtual age of this system?

We start with considering the statistical virtual age discussed in Section 3. The survival functions of a series system of n statistically independent components under the baseline and a more severe environment are

$$\overline{F}_b(t) = \prod_1^n \overline{F}_{bi}(t); \quad \overline{F}_s(t) = \prod_1^n \overline{F}_{bi}(W_i(t)), \tag{14}$$

respectively, where $W_i(t)$ is the scale transformation function for the ith component and we assume that the model (4) holds for every component. Thus, each component has its own statistical virtual age $V_i(t) = W_i(t)$, whereas the virtual age for the system $V(t) = W(t)$ can be obtained from the following equation:

$$F_b(W(t)) = \prod_1^n F_{bi}(W_i(t)) \tag{15}$$

or, equivalently, using Eq. (6):

$$\int_0^{W(t)} \sum_1^n \lambda_{bi}(u)\,\mathrm{d}u = \sum_1^n \int_0^{W_i(t)} \lambda_{bi}(u)\,\mathrm{d}u. \tag{16}$$

Example 4. Let $n = 2$. Assume for simplicity that $W_1 = t$, which means that the first component is protected from the stress (environment), and that the virtual statistical age of the second component is $W_1(t) = 2t$. Therefore the second component has a higher level of degradation. Then Eq. (16) turns to

$$\int_0^{W(t)} (\lambda_{b1}(u) + \lambda_{b2}(u))\,\mathrm{d}u = \int_0^t \lambda_{bi}(u)\,\mathrm{d}u + \int_0^{2t} \lambda_{b2}(u)\,\mathrm{d}u.$$

Assume that the failure rates are linear $\lambda_{b1}(t) = \lambda_1 t$, $\lambda_{b2}(t) = \lambda_2 t$, $\lambda_1, \lambda_2 > 0$. Then

$$W(t) = \left(\sqrt{\frac{\lambda_1 + 4\lambda_2}{\lambda_1 + \lambda_2}}\right) t. \tag{17}$$

If the components are statistically identical in the baseline environment ($\lambda_1 = \lambda_2$), then $W(t) = \sqrt{5/2}t \approx 1.6t$, which means that the virtual age of a system with chronological age t is approximately $1.6t$. It is clear that the 'weight' of components is eventually defined by the relationship between λ_1 and λ_2. When, e.g., λ_1/λ_2 tends to 0, the virtual age of a system tends to $2t$ the virtual age of the second component. On the other hand, if λ_2/λ_1 tends to 0, than the virtual age of a system tends to t the virtual age of the first component.

Going back to the information-based virtual age, as the first choice, we shall weight ages in the series system of n degrading components in accordance with the importance of the components with respect to the failure of the system. Let $V_i(t)$ denote the information-based virtual age of the ith component with a failure rate $\lambda_i(t)$ in a series system of n statistically independent components. Then the virtual age of a system at time t is defined as an expected value of the virtual age of a failed in $[t, t + \mathrm{d}t)$ component:

$$V(t) = \sum_1^n \frac{\lambda_i(t)}{\lambda_s(t)} V_i(t), \tag{18}$$

where $\lambda_s(t) = \sum_1^n \lambda_i(t)$ is the failure rate of the series system.

Thus, the input of each component into the virtual age of a system is weighed in accordance with the value of the corresponding failure rate.

As in the previous section, the second approach is based on the notion of the MRL function. A positive feature of this approach is that unlike (18), we are not forced to link the virtual age with a failure of a system. Similar to (14), the survival function of a series system of n components and the survival function of a remaining lifetime are

$$\overline{F}(x) = \prod_1^n \overline{F}_i(x), \quad \overline{F}(x|t) = \frac{\overline{F}(x+t)}{\overline{F}(x)} = \prod_1^n \overline{F}_i(x|t),$$

respectively. It is convenient for this case to denote the argument in survival functions by x reserving t, as in Eq. (12), for the argument of the MRL function. The corresponding MRL can be obtained, using this definition. Denote now by $F_{I,i}(x,t)$ the information based Cdf of the remaining lifetime for the ith component. Then, in accordance with Section 4, the corresponding MRL and information-based MRL for the series system are defined as

$$M(t) = \int_0^\infty \prod_1^n \overline{F}_i(x|t)\,\mathrm{d}x, \quad M_I(t) = \int_0^\infty \prod_1^n \overline{F}_{I,i}(x,t)\,\mathrm{d}x. \tag{19}$$

and can be calculated numerically for specific settings. Eventually, Eq. (13) should be used for obtaining the corresponding information-based virtual age of a series system in this case. Note, that Eq. (18) defines the explicit weighting of virtual ages with respect to the corresponding failure rates, whereas the second approach performs this weighting implicitly.

Example 5. Consider two systems of the type described in Example 2, connected in series. The first system has parameters λ and n (as in Example 2), and the second μ and m, respectively. It can be shown in this case that the mean time to failure of this series system can be explicitly calculated as

$$M(0) = \frac{1}{\lambda+\mu} \sum_{k=\min(n,m)}^{n+m-1} k \left[\binom{k-1}{n-1} \left(\frac{\lambda}{\lambda+\mu}\right)^n \left(\frac{\mu}{\lambda+\mu}\right)^{k-n} + \binom{k-1}{m-1} \left(\frac{\lambda}{\lambda+\mu}\right)^{k-m} \left(\frac{\mu}{\lambda+\mu}\right)^m \right]. \tag{20}$$

It is clear that:

$$\frac{1}{\lambda+\mu} < M(t) < M(0),$$

where $M(t)$ can be obtained numerically using (19). Assume, that for some time t, our observation is that no standby components are left in both systems. Then, in accordance with Eq. (13), the information-based virtual age of our series system is

$$V(t) = t + \left(M(t) - \frac{1}{\lambda+\mu}\right) > t,$$

which obviously illustrates the fact that this observation indicates the higher level of deterioration than the one that corresponds to the statistical virtual age t.

If observation at time t is $n_I \leqslant n$, $m_I \leqslant n$, then $M_I(t)$ is obtained using Eq. (20), where n and m are substituted by n_I and m_I, respectively.

6. Concluding remarks

A virtual age of a degrading system can be probably considered as some elusive concept, but it certainly makes sense, if properly defined. In this paper we have considered two approaches.

The first one is based on the fact that deterioration depends on the environment. In a more severe environment deterioration is more intensive, which means that objects are aging faster. Therefore, when we want to compare the ages of two systems, using the baseline regime as a reference one, the virtual age of a system, which was functioning under a more severe one, is larger than a chronological age t.

The second approach is based on considering the observed level of individual degradation and comparing it with some average, 'population degradation'.

It should be noted, however, that both approaches are considered under rather stringent, simplifying assumptions, but hopefully our reasoning can be used also for the more general settings in the future.

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References

[1] Kijima M. Some results for repairable systems with general repair. J Appl Prob 1989;26:89–102.
[2] Finkelstein MS. Some notes on two types of minimal repair. Adv Appl Prob 1992;24:226–8.
[3] Finkelstein MS. Modeling a process of non-ideal repair. In: Limnios N, Nikulin M, editors. Recent advances in reliability theory. Birkhauser; 2000. p. 41–53.
[4] Aven T, Jensen U. Stochastic models in reliability. Berlin: Springer; 1999.
[5] Ross SM. Stochastic Processes. New York: Wiley; 1996.
[6] Cox DR, Oakes D. Analysis of survival data. London: Chapman & Hall; 1984.
[7] Finkelstein MS. Wearing-out components in variable environment. Rel Eng Syst Saf 1999;66(N3):235–42.
[8] Hoyland A, Rausand M. System reliability theory. Models and statistical methods. New York: Wiley; 1993.
[9] Finkelstein MS. How does our *n*-component system perform? IEEE Trans Reliab 2001;50:414–8.
[10] Gupta RC, Akman HO. Mean residual life function for certain types of non-monotonic aging. Stochastic models 1995;11:219–25.
[11] Barlow R, Proschan F. Statistical theory of reliability and life testing. Probability models. New-York: Holt, Rinehart and Winston; 1975.