BOUNDS FOR THE FAILURE RATE IN HETEROGENEOUS POPULATIONS

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Summary: We derive bounds for the mixture failure rate in the multiplicative (proportional hazards) model. We explicitly show that the corresponding proportionality in subpopulations does not result in proportionality for a whole population. We analyze the shape of the mixture failure rate in an environment with a stress change point and also discuss the effect of shocks which change a mixing distribution.

1. Introduction

One can hardly find homogeneous populations in real life, although most of the studies on the failure rate modelling deal with a homogeneous case. Neglecting existing heterogeneity can lead to substantial errors in stochastic analysis in reliability, survival and risk analysis and other disciplines.

Mixtures of distributions usually present an effective tool for modelling heterogeneity. It is well known that mixtures of decreasing failure rate (DFR) distributions are always characterized by the DFR (Barlow and Proschan,

MSC:

1975). On the other hand, mixtures of increasing failure rate distributions (IFR) can decrease at least in some intervals of time, which means that the IFR class of distributions is not closed under the operation of mixing (Lynch, 1999).

A natural approach to the mixture modelling exploits a notion of a non-negative, random, unobserved parameter (frailty) Z introduced by Vaupel et al. (1979) for the Gamma-distributed Z. In fact, this approach results in considering a random failure rate $\lambda(t, Z)$. As the failure rate is a conditional characteristic, the 'ordinary' expectation $E[\lambda(t, Z)]$ with respect to Z does not define a mixture failure rate $\lambda_m(t)$ and a proper conditioning should be performed (Finkelstein, 2004).

The shape of the mixture failure rate is important in many applications and has been studied intensively in the literature. For instance, Gurland and Sethuraman (1994) considered some examples of mixtures that have strictly decreasing failure rates, although each of the distributions that are being mixed has a non-decreasing failure rate. In Block et al. (2003) it was shown that the failure rate of a mixture of two distributions with linearly increasing failure rates can exhibit a rather bizarre behaviour: there can be up to four (!) changes in monotonicity. As mixture failure rates are often DFR, at least in the initial interval of time $[0, t']$, $t' > 0$, the corresponding burn-in procedures can be implemented for the heterogeneous population of engineering components (Block and Savits, 1997). Navarro and Hernandez (2004) have developed interesting techniques for obtaining the bathtub shaped failure rates from mixtures of two positive truncated normal distributions.

Considerable attention has been given to the study of asymptotic behaviour of mixture failure rates as $t \to \infty$. In Block et al. (1993) it was proved, that if the failure rate of each subpopulation converges to a constant and this convergence is uniform, then the mixture failure rate converges to the failure rate of the strongest subpopulation: the weakest sub-populations are dying out first. In his recent paper Li (2005) generalizes these results. Note that analytical restrictions in these findings, e.g. uniform convergence, are rather stringent. Finkelstein and Esaulova (2006a) avoid these restrictions and consider the problem from a different viewpoint exploiting the technique of the Abelian-type theorems. The class of considered survival models is rather general and contains proportional hazards and accelerative life models as specific cases.

In the current paper we are mostly interested in simple bounds for the mixture failure rate for the multiplicative model of mixing. The obtained bounds can be very helpful in various applications, e.g. for mortality analysis in heterogeneous populations (Thatcher, 1999). Specifically, we show that when subpopulations obey the PH model with the multiplicative frailty Z and the common proportionality factor k the resulting mixture failure rate has a strict upper bound $k\lambda_m(t)$, where $\lambda_m(t)$ has a meaning of a mixture failure rate in a heterogeneous population without a proportionality factor $(k \equiv 1)$. Furthermore, this result presents another explicit justification of the fact that the proportional hazards (PH) model in each realization does not result in the PH model for the corresponding mixture failure rates. An important applications case of a step-stress changepoint in the proportional hazards framework is also considered here and the corresponding bounds for the mixture failure rate are derived. Another example deals with a special type of shock, which performs a burn-in for heterogeneous populations.

A similar technique has been used in Finkelstein and Esaulova (2006b) for ordering mixture failure rates with different mixing distributions, whereas in the current paper we are focused on obtaining bounds for $\lambda_m(t)$ in different settings.

2. Bounds for the mixture failure rate in a PH model

Let $T \geq 0$ be a lifetime random variable with the Cdf $F(t)\left(\overline{F}(t) \equiv 1 - F(t)\right)$. Assume that $F(t)$ is indexed by a random variable Z:

$$P(T \leq t \mid Z = z) \equiv P(T \leq t \mid z) = F(t, z)$$

and that the pdf $f(t, z)$ exists. Then the corresponding failure rate $\lambda(t, z)$ is $f(t, z)/\overline{F}(t, z)$. Let Z be interpreted as a non-negative random variable with support in $[0, \infty)$ and the pdf $\pi(z)$. Thus, a mixture Cdf is defined by

$$F_m(t) = \int_0^\infty F(t, z)\,\pi(z)\,dz.$$

As the failure rate is a conditional characteristic, the mixture failure rate $\lambda_m(t)$ should be defined in the following way (see, e.g., Shaked and Spizzichino, 2001; Finkelstein and Esaulova, 2001 to name a few):

$$\lambda_m(t) = \frac{\int_0^\infty f(t, z)\,\pi(z)\,dz}{\int_0^\infty \overline{F(t, z)\,\pi(z)\,dz}} = \int_0^\infty \lambda(t, z)\,\pi(z \mid t)\,dz, \qquad (1)$$

where

$$\pi(z \mid t) \equiv \pi(z)\,\frac{\overline{F}(t, z)}{\int_0^\infty \overline{F(t, z)\,\pi(z)\,dz}}.$$

Consider now a specific multiplicative frailty model:

$$\lambda(t, z) = z\lambda(t) \qquad (2)$$

where $\lambda(t)$ is the baseline failure rate. It can be shown easily that due to the multiplicative nature of the model, equation (1) can now be written as

$$\lambda_m(t) = \frac{(L_\pi(\Lambda(t)))'}{L_\pi(\Lambda(t))} = -(\log(L_\pi(\Lambda(t))))', \tag{3}$$

where

$$L_\pi(s) = E[\exp\{-sZ\}]$$

is the Laplace transform of the mixing distribution $\pi(z)$ and $\Lambda(t) = \int_0^t \lambda(u)\,du$ is a cumulative baseline failure rate, whereas notation $(f(g(t)))'$ means $\dfrac{d(f(g(t)))}{d(g(t))}$.

Formally combine model (2) with a PH model in the following way:

$$\lambda(t,\ z,\ k) = zk\lambda(t) \equiv z_k\lambda(t). \tag{4}$$

Therefore, the baseline $F(t)$ is indexed by the random variable $Z_k = kZ$ with the pdf $\pi_k(z) = \pi(z\,/\,k)$. Equivalently, (4) can be interpreted as a frailty model with a mixing random variable Z and the baseline failure rate $k\lambda(t)$. These two simple equivalent interpretations will help us in what follows. Assume for simplicity that $E[Z] < \infty$. Substitution of equations (2) and (4) into (1) gives

$$\lambda_m(t) = \lambda(t)\int_0^\infty z\pi(z \mid t)\,dz \equiv \lambda(t)\,E[Z \mid t] \tag{5}$$

$$\lambda_{mk}(t) = k\lambda(t)\int_0^\infty z\pi_k(z \mid t)\,dz \equiv \lambda(t)\,E[Z_k \mid t]. \tag{6}$$

As $Z_k = kZ$, its pdf is

$$p_k(k) = \frac{1}{k}\pi\left(\frac{z}{k}\right).$$

Theorem 1 Let the mixture failure rates for the multiplicative models (2) and (4) be given by relations (5) and (6), respectively, where $k > 1$.

Assume that the following quotient increases in z:

$$\frac{\pi_k(z)}{\pi(z)} = \frac{\pi\left(\frac{z}{k}\right)}{k\pi(z)} \uparrow \tag{7}$$

Then

$$\lambda_{mk}(t) > \lambda_m(t)\,; \;\; \forall t \in [0,\ \infty). \tag{8}$$

Proof Although inequality (8) seems rather trivial at first glance, it is valid only for some specific cases of mixing (e.g., the multiplicative model). It is clear that (8) is always true for sufficiently small t, whereas for larger values of time the ordering can be different for general mixing models. Denote:

$$\Delta\lambda_m(t) = \lambda_{mk}(t) - \lambda_m(t)\,.$$

Using definition (1), it can be seen that the sign of this difference is defined by the sign of

$$\begin{aligned}
&\int_0^\infty z\overline{F}(t,\ z)\,\pi_k(z)\,dz \int_0^\infty \overline{F}(t,\ z)\,\pi(z)\,dz \\
&-\int_0^\infty z\overline{F}(t,\ z)\,\pi_k(z)\,dz \int_0^\infty \overline{F}(t,\ z)\,\pi(z)\,dz \\
=\ &\int_0^\infty\int_0^\infty \overline{F}(t,\ u)\,\overline{F}(t,\ s)\left[u\pi_k(u)\pi(s) - s\pi_k(u)\,\pi(s)\right] duds \\
=\ &\underset{u>s}{\int_0^\infty\int_0^\infty} \overline{F}(t,\ u)\,\overline{F}(t,\ s)\left[\pi_k(u)\,\pi(s)\,(u-s) + \pi_k(s)\,\pi(u)\,(s-u)\right] duds \\
=\ &\underset{u>s}{\int_0^\infty\int_0^\infty} \overline{F}(t,\ u)\,\overline{F}(t,\ s)\,(u-s)\left(\pi_k(u)\,\pi(s) - \pi_k(s)\,\pi(u)\right) duds.
\end{aligned} \tag{9}$$

Therefore, the sufficient condition for inequality (8) is condition (7). It is easy to verify that this condition is satisfied, e.g. for the Gamma and the Weibull densities which are often used for mixing.

Example 1 Consider the frailty model (3), where Z has a gamma distribution:

$$\pi(z) = \frac{\beta^{a}}{\Gamma(\alpha)} z^{\alpha-1} \exp\{-\beta z\}\,; \quad \alpha > 0,\ \beta > 0.$$

As was mentioned, condition (7) is satisfied in this case and therefore inequality (8) takes place. But this can also be shown by direct integration. Substituting into relation (1):

$$\lambda_m(t) = \frac{\lambda(t)\int_0^\infty \exp\{-z\Lambda(t)\} z\pi(z)\,dz}{\int_0^\infty \exp\{-z\Lambda(t)\}\pi(z)\,dz}.$$

Computation of integrals results in

$$\lambda_m(t) = \frac{\alpha\lambda(t)}{\beta + \Lambda(t)}. \tag{11}$$

Equation (11) can now be written in terms of $E[Z]$ and $Var(Z)$:

$$\lambda_m(t) = \lambda(t)\frac{E^2[Z]}{E[Z] + Var(Z)\Lambda(t)}, \tag{12}$$

which for the specific case $E[Z] = 1$ gives the result of Vaupel et al. (1979) which is widely used in demography :

$$\lambda_m(t) = \frac{\lambda(t)}{1 + Var(Z)\Lambda(t)}. \tag{13}$$

On the other hand, considering model (4) for $k > 1$ results in

$$\begin{aligned}\lambda_{mk}(t) &= \lambda(t)\frac{E^2[Z_k]}{E[Z_k] + Var(Z_k)\Lambda(t)} \\ &= \lambda(t)\frac{k^2E^2[Z]}{kE[Z] + k^2Var(Z)\Lambda(t)} > \lambda_m(t),\end{aligned}$$

which is a direct proof of inequality (8) for this specific case.

Now we shall obtain an upper bound for $\lambda_{mk}(t)$.

Theorem 2 Let the mixture failure rates for the multiplicative models (3) and (4) be given by relations (5) and (6), respectively, where $k > 1$.
Then

$$\lambda_{mk}(t) < k\lambda_m(t)\,; \;\; \forall t \in (0,\ \infty)\,. \tag{14}$$

Proof Consider the difference $\lambda_{mk}(t) - k\lambda_m(t)$ similar to (9), but in a slightly different way: $\lambda_{mk}(t)$ will be equivalently defined by the baseline failure rate $k\lambda(t)$ and the mixing variable Z (in (9) it was defined by the baseline $\lambda(t)$ and the mixing variable kZ). This means that

$$\lambda_{mk}(t) - k\lambda_m(t) = k\lambda(t)\left(\widehat{E}[Z \mid t] - E[Z \mid t]\right), \tag{15}$$

where conditioning in $\widehat{E}[Z \mid t]$ is different from the one in $E[Z \mid t]$ in the described sense. Denote:

$$\overline{F}_k(t,\ z) = \exp\{-zk\Lambda(t)\}\,.$$

'Symmetrically' to (9), *sign*$[\lambda_{mk}(t) - k\lambda_m(t)]$ is defined by

$$sign \int_0^\infty \int_0^\infty \pi(u)\,\pi(s)\,(u-s)\left(\overline{F}_k(t,\ u)\,\overline{F}(t,\ s) - \overline{F}(t,\ u)\,\overline{F}_k(t,\ s)\right) duds,$$

which is negative for all $t > 0$ as

$$\frac{\overline{F}_k(t,\ z)}{\overline{F}(t,\ z)} = \exp\{-(k-1)\,z\Lambda(t)\}$$

is decreasing in z.♦

It is worth noting that we do not need an additional condition for this bound as in the case of Theorem 1. Also it is clear that $\lambda_{mk}(0) = k\lambda_m(0)$. As it was already mentioned, model (4) defines a combination of a PH and a frailty model. When $Z = 1$, it is an 'ordinary' PH model. In the presence of a random Z, as follows from (14), the observed failure rate $\lambda_{mk}(t)$ cannot be obtained as $k\lambda_m(t)$ due to the nature of mixing. Therefore, this theorem gives another explicit justification of the fact that *the PH model in* *each realization does not result in the PH model for the corresponding mixture failure rates*.

Example 1 can be continued to illustrate inequality (14):

$$\begin{aligned}\lambda_{mk}(t) &= \lambda(t)\frac{k^2E^2[Z]}{kE[Z]+k^2Var(Z)\Lambda(t)} \\ &< \lambda(t)\frac{kE^2[Z]}{E[Z]+Var(Z)\Lambda(t)} = k\lambda_m(t).\end{aligned}$$

Example 2: **Stable frailty distributions and proportionality**

It follows from Hougaard (2000), that the Laplace transform of a stable distribution is given by

$$L(s) = \exp\left\{-\frac{\beta s^\alpha}{\alpha}\right\},$$

where β is a positive parameter and $\alpha \in (0,\ 1]$ for positive stable distributions. Note that the value $\alpha = 1$ corresponds to the degenerate frailty distribution and therefore is not considered. Applying equation (3) to initial model (2) results in

$$\lambda_m(t) = \beta\lambda(t)(\Lambda(t))^{\alpha-1}.$$

On the other hand, applying equation (3) to model (4) gives

$$\lambda_{mk}(t) = k^\alpha\beta\lambda(t)(\Lambda(t))^{\alpha-1} = k^\alpha\lambda_m(t).$$

Therefore, we observe proportionality in this setting, but with the changing coefficient of proportionality (from k to k^α , respectively).

It is clear that this specific result does not contradict our theorems, as it follows directly from these equations that for positive stable distributions $(\alpha \in (0,\ 1))$ and $k > 1$:

$$\lambda_m(t) < \lambda_{mk}(t) < k\lambda_m(t);\ \ \forall\, t \in (0.\ \infty),$$

which are, in fact, inequalities (8) and (14).

3. Impact of environment on mixing distributions

Changes in time in mixing distributions can occur due to various reasons. In this section we shall consider two specific cases.

3.1 Change in environment (stress)

Assume that there are two possible environments (stresses): $\varepsilon(t)$ and $\varepsilon_s(t)$ – the baseline and a more severe one, respectively. The baseline environment for our heterogeneous population corresponds to the observed failure rate $\lambda_m(t)$ and a more severe one to $\lambda_{mk}(t)$, $k > 1$. As previously, assume also that the PH model holds for each subpopulation (for each fixed z). Consider a piece-wise constant step stress with a single change point at t_1:

$$\varepsilon(t_1) = \begin{cases} \varepsilon, & 0 \le t < t_1 \\ \varepsilon_k & t \ge t_1 \end{cases} \tag{16}$$

where the stresses ε and ε_k correspond to the failure rates $z\lambda(t)$ and $zk\lambda(t)$, respectively $(k > 1,\ z \ge 0)$. In accordance with a 'memory-less property' of the PH model, the stress (16) results in the following failure rate:

$$\lambda(t,\ t_1,\ z,\ k) = \begin{cases} z\lambda(t), & 0 \le t < t_1 \\ kz\lambda(t) & t \ge t_1 \end{cases} \tag{17}$$

for each subpopulation.

Denote the resulting mixture failure rate in this case as

$$\lambda_m(t,\ t_1) = \begin{cases} \lambda_m(t), & 0 \le t < t_1 \\ \widetilde{\lambda}_{mk}(t) & t \ge t_1 \end{cases} \tag{18}$$

where similar to the previous section, where $\lambda_{mk}(0) = k\lambda_m(0)$, the analogous equation holds for the change point at t_1:

$$\widetilde{\lambda}_{mk}(t_1) = k\lambda_m(t_1). \tag{19}$$

Relation (19) means that the model with a step stress is proportional for the mixture failure rates only at t_1.

We want to prove the following inequality:

$$\lambda_{mk}(t) < \widetilde{\lambda}_{mk}(t)\,;\ \ \forall\, t \in [t_1,\ \infty) \tag{20}$$

In accordance with (18), consider two initial (for the interval $[t_1,\ \infty)$) mixing random variables: $Z_1 = Z \mid T_1 > t_1$, where T_1 is defined by the baseline failure rate $k\lambda(t)$ (see interpretation of relations (2) and (4)) and $\widetilde{Z}_1 = Z \mid \widetilde{T}_1 > t_1$, where $\widetilde{T}_1$ is defined by the baseline failure rate $\lambda(t)$. As follows from definition (1), the ratio of the corresponding densities

$$\frac{\widetilde{\pi}(z,\ t_1)}{\pi(z,\ t_1)} = \exp\{(k-1)\, z\Lambda(t_1)\}$$

is increasing in z, therefore condition (7) holds and inequality (20) follows immediately from (9) with obvious alterations caused by the change in the left end point of an interval from 0 to t_1.

Example 3 We illustrate inequality (20) by an example from Vaupel and Yashin (1985). Consider a discrete mixture of two subpopulations of humans with the Gompertz baseline failure rate. Thus, an entire cohort consists of a frail and a robust cohort with mortality rates $\mu_j(t)$ and $\mu_r(t)$, respectively. Assume that the health progress reduces mortality rate proportionally to $\widetilde{\mu}_{\int}(t)$ and $\widetilde{\mu}_r(t)$, respectively:

$$\mu_{\int}(t) = k\widetilde{\mu}_{\int}(t)\,;\quad \mu_r(t) = k\widetilde{\mu}_r(t)\,,\quad k > 1.$$

It follows from (8) that the corresponding mixture mortality rates for the two regimes are ordered as

$$\widetilde{\mu}_m(t) < \mu_m(t)\,;\ \ \forall\, t \in [0,\ \infty).$$

Now, in accordance with the setting of this section, assume that the health progress reduces mortality rates in subpopulations only at younger ages $([0,\ t_1))$ and leaves them unchanged in $(t_1,\ \infty)$. Then, as follows from (20):

$$\mu_m(t) < \widetilde{\mu}_m(t)\,;\ \ \forall\, t \in [t_1,\ \infty).$$

Thus, in $[0,\ t_1)$ mixture mortality rate is evidently reduced, but strangely enough this early life reduction increases mixture mortality rate in $[t_1,\ \infty)$ compared with a 'no reduction' case. "Every individual's life chances are improved at younger ages and are as good as ever at later ages, but observed (mixture) cohort mortality makes it look as if pediatricians are making progress whereas gerontologists are losing ground" (Vaupel and Yashin (1985). These authors have used simulations to illustrate the phenomenon, whereas we have proved it analytically.

3.2 Shocks in heterogeneous populations

Consider now a general mixing model (1) and assume that at time $t = t_1$ an instantaneous shock had occurred, which affects the whole population: with the corresponding complementary probabilities it either kills an individual, or 'leaves him unchanged'. Without losing generality, let $t_1 = 0$, otherwise a new initial mixing variable $Z \mid t_1$ should be defined and the corresponding procedure can easily be adjusted to this case. It is natural to suppose that the more frail (with larger failure rate) individuals are, the more susceptible they are to dying.

This setting can be defined probabilistically in the following way: Let $\pi_1(z)$ denote a frailty distribution of a random variable Z_1 after a shock and let $\lambda_{ms}(t)$ be the corresponding observed (mixture) failure rate after it. Assume that

$$\pi_1(z) = \frac{g(z)\,\pi(z)}{\int_{\alpha}^{b} g(z)\,\pi(z)\,dz} \tag{21}$$

where $g(z)$ is a decreasing function and therefore $\pi_1(z)/\pi(z)$ is decreasing. It means that a shock performs a kind of a burn-in operation (Block et al., 1993) and Z and Z_1 are ordered in the sense of the likelihood ratio (Ross, 1996;

Shaked and Shanthikumar, 1993):

$$Z \geq_{LR} \quad Z_1 \tag{22}$$

Now we are able to formulate the following result:

Theorem 3 Let relation (21), defining a mixing density after a shock at $t = 0$, where $g(z)$ is a decreasing function, hold.

Assume that

$$\lambda(t,\ z_1) < \lambda(t,\ z_2),\ \ z_1 < z_2,\ \ \forall\, z_1,\ z_2 \in [0,\ \infty],\ \ t \geq 0. \tag{23}$$

Then

$$\lambda_{ms}(t) < \lambda_m(t);\ \ \forall\, t \in [0,\ \infty). \tag{24}$$

Proof Inequality (23) is a natural ordering in the family of failure rates $\lambda(t,\ z),\ \ z \in [0,\ \infty)$ and trivially holds for the specific model (2). Conducting all steps as when obtaining relation (9), finally results in

$$\begin{aligned} & \textit{sign}\left[\lambda_{ms}(t) - \lambda_m(t)\right] \\ = \ & \textit{sign} \int\limits_{\substack{\alpha \\ u>s}}^{b} \int\limits_{\alpha}^{b} \overline{F}(t,\ u)\,\overline{F}(t,\ s)\,(\lambda(t,\ u) - \lambda(t,\ s)) \\ & \cdot (\pi_1(u)\,\pi(s) - \pi_1(s)\,\pi(u))\,\textit{duds}, \end{aligned}$$

which is negative due to definition (21) and assumptions of this theorem.⧫

At $t = 0$, for instance

$$\lambda_m(0) - \lambda_{ms}(0) = \int\limits_0^{\infty} \lambda(0,\ z)\,(\pi(z) - \pi_1(z))\,dz.$$

In accordance with inequality (24), the curve $\lambda_{ms}(t)$ lies beneath the curve $\lambda_m(t)$ for $t \geq 0$. This fact seems intuitively evident, but, in fact, it is valid only due to the rather stringent conditions of this theorem. It can be shown, for instance, that the replacement of condition (22) by a weaker one of stochastic dominance, $Z \geq_{st} Z_1$, will not guarantee ordering (24) for all t.

4. Concluding remarks

We derive bounds for the mixture failure rate of heterogeneous populations and show analytically that a PH model, which holds for subpopulations, is violated for the observed (mixture) failure rate. In Example 2 we consider the case of positive stable mixing distributions, when proportionality holds but with a different coefficient.

Shocks with described stochastic properties 'push down' the initial mixture failure rate (ordering (24)). It will be interesting to consider other models for a shock's impact on a heterogeneous population, e.g. when the function $g(z)$ in (21) is increasing or nonmonotone.

Changes in environment can lead to surprising effects in mixture failure rate dynamics. Example 3 shows this phenomenon in the demographic context.

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