Available online at www.sciencedirect.com
ScienceDirect
Reliability Engineering and System Safety 92 (2007) 1671–1676
RELIABILITY ENGINEERING & SYSTEM SAFETY
www.elsevier.com/locate/ress

Imperfect repair and lifesaving in heterogeneous populations

Maxim Finkelstein[a,b,*]

[a]Department of Mathematical Statistics, University of the Free State, PO Box 339, 9300 Bloemfontein, South Africa
[b]Max Planck Institute for Demographic Research, Rostock, Germany

Accepted 22 September 2006
Available online 4 December 2006

Abstract

In this theoretical paper we generalize the notion of minimal repair to the heterogeneous case, when the lifetime distribution function can be modeled by continuous or a discrete mixture of distributions. The statistical (black box) minimal repair and the minimal repair based on information just before the failure of an object are considered. The corresponding failure (intensity) rate processes are defined and analyzed. Demographic lifesaving model is also considered: each life is saved (cured) with some probability (or equivalently a proportion of individuals who would have died are now resuscitated and given another chance). Those who are saved experience the *statistical* minimal repair. Both of these models are based on the Poisson or non-homogeneous Poisson processes of underlying events, which allow for considering heterogeneity. We also consider the new model of imperfect repair in the homogeneous case and present generalizations to the heterogeneous setting.

Keywords: Imperfect repair; Minimal repair; Poisson process; Lifesaving model; Repairable systems

1. Introduction

Most of the papers on reliability and survival analysis deal with the homogeneous case although one can hardly find homogeneous populations in practice. Heterogeneity can introduce new and sometimes unexpected features in reliability analysis when compared with a homogeneous case. It is well known that the (observed) failure rate in a heterogeneous population does not follow the pattern of the subpopulations failure rates. For instance, the mortality rate of a homogeneous human population is approximately an exponentially increasing function (Gompertz law), but recently the deceleration of this increase for old ages was observed. A natural explanation of this fact lies in the population heterogeneity.

In conventional reliability analysis of repairable systems, it was also always assumed that objects to be repaired 'are chosen' from a homogeneous population. It turns out, that generalization to the heterogeneous case is straightforward only for the perfect repair. In this case, the process of functioning of an instantaneously repaired item can be, as usually, modeled by the renewal process with a corresponding mixture as a distribution of inter-arrival times.

Even the simplest type of imperfect repair-a minimal repair, complicates modeling to a great extent, which is shown in Sections 2 and 3. In Section 4, we introduce a new imperfect repair model based on the proportional impact of the repair action on the corresponding failure rate and generalize this approach to the heterogeneous setting. In Section 5 the heterogeneous lifesaving model is considered. This model is also based on the notion of minimal repair and can be considered as the corresponding application.

This is a theoretical note, but we believe that the results modeling an impact of heterogeneity on reliability characteristics of real objects can be important in various applications. They show specifically that this impact should be taken into account in reliability analysis of repairable systems.

2. Homogeneous case

Consider an object with an absolutely continuous time to failure cumulative density function (Cdf) $F(t)$ and a failure

*Corresponding author. Department of Mathematical Statistics, University of the Free State, PO Box 339, 9300 Bloemfontein, South Africa.
E-mail address: FinkelM.SCl@mail.uovs.ac.za.

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doi:10.1016/j.ress.2006.09.018

rate $\lambda(t)$, which starts operating at $t=0$. Assume that the repair action is performed instantaneously upon failure. The repair is usually qualified as *perfect*, if the Cdf of the repaired object is $F(t)$ (as good as new) and as the *minimal* repair at time x, if its Cdf is:

$$F(t|x) \equiv 1 - \frac{1-F(t+x)}{1-F(x)}. \tag{1}$$

It is clear that the *minimal* repair does not change the failure rate of our object. This type of minimal repair, when the only information at hand is the age x of the failed unit is called also the *statistical* (*or black box*) *minimal* repair. On the other hand, sometimes we can observe on failure some additional information, defining the state of an object (e.g. the structure of a system). This can result in a more general type of repair, which is usually called the *information-based* (*or physical*) *minimal* repair. The *information-based minimal* repair brings our object back to the state (to be defined by the relevant information) it had just before the failure [1,3,6].

A convenient mathematical description of repair processes uses a concept of the stochastic (or failure) intensity [4]. Let, firstly, T_n, $n=12,\ldots$ be some general orderly point process defined on the basic probability space, which can be interpreted as instances of repairs (or, equivalently, instantaneously repaired failures). Repairs can be perfect, minimal or imperfect [7]. Denote by N_t, $t \geqslant 0$ the corresponding counting process. The following decomposition exists under some mild assumptions:

$$N_t = \int_0^t \lambda_u \,\mathrm{d}u + M_t, \tag{2}$$

where λ_t is the corresponding stochastic intensity and M_t is a martingale with a zero mean [4]. The classical example of λ_t is defined by the renewal process (perfect repair) with the monotone increasing failure rate $\lambda(t)$ of the underlying Cdf $F(t)$:

$$\lambda_t = \sum_{n=0}^{\infty} \lambda(t-T_n) I(T_n \leqslant t < T_{n+1}), \quad T_0 = 0. \tag{3}$$

Another standard example is the 'deterministic stochastic intensity' $\lambda_t = \lambda(t)$, which defines the non-homogeneous Poisson process (NHPP) of repairs with intensity $\lambda(t)$. It is well known that, in accordance with definition (1), this example can be also interpreted as the process of *statistical minimal* repairs. Generalizing this notion, Aven and Jensen [3] suggested to define a point process as the process of *F-minimal* repairs, if one cannot define the failure time points from observation of λ_t. Obviously, according to this definition, the Poisson process is the process of *F-minimal* repairs, whereas the renewal process is not.

Thus, the *F-minimal* repair is the specific case of the *information-based minimal* repair. Minimal repair process for heterogeneous populations to be considered in this paper is an example, when the failure time points *can be defined* from observation of λ_t.

The following minimal repair model has important demographic and biological applications. Let $\lambda(t)$ be an 'ordinary' failure rate for a homogeneous population. Assume the following lifesaving procedure: each life, characterized by initial failure rate $\lambda(t)$ is saved (repaired) with probability $1-\theta(t)$ (or equivalently a proportion of individuals who would have died are now resuscitated and given another chance). Those who are saved, experience the (statistical) minimal repair. The number of resuscitations (repairs) is unlimited. In fact, this is a partially minimal repair model, as a proportion of individuals are not saved (not repaired). Under these assumptions, it was proved analytically in [13] that the described lifesaving procedure results in the new failure rate

$$\lambda_r(t) = \theta(t)\lambda(t). \tag{4}$$

The similar result was obtained for different reliability related settings in [5,7]. The following shock model [7] gives another useful interpretation and, in fact, presents a more general non-technical proof. Consider an object subject to a general orderly stochastic point process of shocks. Assume that a shock, affecting an object at time $t \in (0, \infty)$, independently of the previous shocks, causes a failure (death) with probability $\theta(t)$ and does not cause any changes in the object with a complementary probability $1-\theta(t)$. Assume for simplicity that this is the only cause of failure of an object. Let $\lambda_c(t, H_t)$ denote the corresponding complete intensity function, where H_t is a history of the process up to and including t. The useful interpretation is that $\lambda_c(t, H_t)\,\mathrm{d}t + o(\mathrm{d}t)$ defines the probability of a shock in $(t, t+\mathrm{d}t]$ given the history H_t (the configuration of shocks in $(0,t]$. The complete intensity function is an alternative (to λ_t) way of defining the point process. The conditional hazard can be defined in this case by the following probability:

$$P\{T \in (t, t+\mathrm{d}t] | H_t, T > t\} = \theta(t)\lambda_c(t, H_t)\,\mathrm{d}t. \tag{5}$$

Assume now that shocks occur in accordance with the non-homogeneous Poisson process $P_t t \geqslant 0$ with intensity $\lambda(t)$. It is clear that in this case

$$\lambda_c(t, H_t) \equiv \lambda(t).$$

Then relation (5) reads

$$P\{T \in (t, t+\mathrm{d}t] | H_t, T > t\} = \theta(t)\lambda(t)\,\mathrm{d}t \tag{6}$$

and, finally, the corresponding survival probability is given by

$$\begin{aligned}\bar{F}(t) &= \exp\left\{-\int_0^t \theta(u)\lambda(u)\,\mathrm{d}u\right\} \\ &= \exp\left\{-\int_0^t \lambda_r(u)\,\mathrm{d}u\right\}.\end{aligned} \tag{7}$$

Eq. (7) obviously follows from the fact that the conditional hazard $\lambda_c(t, H_t)$ does not depend on history H_t.

3. Heterogeneous minimal repair model

Let $T \geqslant 0$ be a lifetime random variable (r.v.) with the Cdf $F(t)$. As usually, denote the survival function by $\bar{F}(t)$. Assume that $F(t)$ is indexed by a r.v. Z: $P(T \leqslant t|Z=z) \equiv P(T \leqslant t|z) = F(t,z)$ and that the probability density function (pdf) $f(t,z)$ exists. Then the corresponding failure rate $\lambda(t,z)$ is defined by $f(t,z)/\bar{F}(t,z)$. Let Z be interpreted as a non-negative r.v. with support in $[a,b]$, $a \geqslant 0$, $b \leqslant \infty$ and the pdf $\pi(z)$.

Another meaningful interpretation defines an unobserved Z as a frailty in the heterogeneous population [12]. Therefore, the mixture models can be alternatively called the frailty models. The above setting leads naturally to considering mixtures of distributions, which are useful for describing heterogeneity:

$$F_m(t) = \int_a^b F(t,z)\pi(z)\,\mathrm{d}z \tag{8}$$

whereas the mixture failure rate in accordance with the definition (e.g. [8]) is

$$\lambda_m(t) = \frac{\int_a^b f(t,z)\pi(z)\,\mathrm{d}z}{\int_a^b \bar{F}(t,z)\pi(z)\,\mathrm{d}z} = \int_a^b \lambda(t,z)\pi(z|t)\,\mathrm{d}z, \tag{9}$$

where the conditional pdf (on condition that $T > t$) is

$$\pi(z|t) \equiv \pi(z|T>t) = \pi(z)\frac{\bar{F}(t,z)}{\int_a^b \bar{F}(t,z)\pi(z)\,\mathrm{d}z}. \tag{10}$$

Consider an object with the Cdf (8) describing a lifetime in a heterogeneous population. Let $T_1 = t_1$ be the realization of time to the first failure (repair). Then the statistical minimal repair is obviously defined by relation (1), where $F(t)$ is substituted by $F_\mathrm{m}(t)$ and x by t_1, whereas the process of minimal repairs of this kind is NHPP with intensity $\lambda_\mathrm{m}(t)$.

It is much more interesting to define the *information-based minimal* repair for the heterogeneous setting [9]. In accordance with the general definition of the *information-based minimal* repair, an object is restored to the state it had just prior the failure. It is reasonable to assume in this case that the state is defined by the frailty Z, which in its tern defines the failure rate of a subpopulation $\lambda(t,z)$. It is clear that, as we observe only the failures T_i, $i=1,2,\ldots$, the stochastic intensity in $[0,t_1)$ is also $\lambda_\mathrm{m}(t) \equiv \lambda_\mathrm{m}^1(t)$, defined by Eq. (10). As unobserved $Z=z$ 'was chosen' at $t=0$ for the future performance of our object, the *information-based minimal* repair restores it to the state defined by $Z=z$. This means that the stochastic intensity in $[t_1,t_2)$ is

$$\lambda_\mathrm{m}^2(t) = \int_a^b \lambda(t,z)\pi(z|t-t_1)\,\mathrm{d}z, \tag{11}$$

where the pdf $\pi(z|t-t_1)$ is given by the adjusted relation (10):

$$\pi(z|t-t_1) = \pi(z)\frac{\bar{F}(t-t_1,z)}{\int_a^b \bar{F}(t-t_1,z)\pi(z)\,\mathrm{d}z}. \tag{12}$$

In accordance with relations (10)–(12) the corresponding stochastic intensity in $[0,\infty)$ is (compare with Eq. (3))

$$\lambda_t = \sum_{n=1}^{\infty} \lambda_\mathrm{m}^n(t) I(T_{n-1} \leq t < T_n), \quad T_0 = 0, \tag{13}$$

where

$$\lambda_\mathrm{m}^n(t) = \int_a^b \lambda(t,z)\pi(z|t-T_{n-1})\,\mathrm{d}z. \tag{14}$$

An interesting feature of (14) is that at failure points it equals the unconditional mean of $\lambda(t,Z)$:

$$\lambda_\mathrm{m}^n(t_n) = \int_a^b \lambda(t_n,z)\pi(z)\,dz, \quad n \geqslant 1. \tag{15}$$

Therefore, the function

$$\lambda_\mathrm{P}(t) = \int_a^b \lambda(t,z)\pi(z)\,\mathrm{d}z \tag{16}$$

is important for describing the model under investigation. The subscript "P" stands for "Poisson", as Eq. (16) defines the mean intensity function of the conditional Poisson process. The corresponding stochastic intensity in this case is defined trivially by

$$\lambda_t = \lambda(t,Z), \quad t \geqslant 0. \tag{17}$$

Model (16)–(17) can be interpreted by considering the observed r.v. Z. In this case for each realization of Z, each failure is minimally (statistically) repaired and the intensity function of the resulting point process is an 'ordinary' mixture $\lambda_\mathrm{P}(t)$.

This model defines a doubly stochastic Poisson process (see also Section 5), which was studied in connection with the minimal repair in some early papers devoted to optimal replacement strategies (see, e.g., [2] and references therein).

It is interesting to compare $\lambda_\mathrm{m}(t)$,$\lambda_\mathrm{P}(t)$ and λ_t. The following example will help us to do so.

Example 1. Let $F(t,z)$ be for simplicity an exponential distribution with parameter $\lambda(t,z) = z\lambda$ and $\pi(z)$ be an exponential pdf in $[0,\infty)$ with parameter ϑ. Using relation (9)

$$\lambda_\mathrm{m}(t) = \frac{\lambda}{\lambda t + \vartheta},$$

where $\lambda_\mathrm{P}(t) = \lambda/\vartheta$ and

$$\lambda_t = \sum_{n=1}^{\infty} \frac{\lambda}{\lambda(t-T_{n-1}) + \vartheta} I(T_{n-1} \leqslant t < T_n), \quad T_0 = 0.$$

Thus,

$$\lambda_\mathrm{m}(t) \leqslant \lambda_t \leqslant \lambda_P(t), \quad t > 0 \tag{18}$$

and $\lambda_t = \lambda_\mathrm{P}(t)$ only at random failure points T_n, $n \geqslant 1$ whereas $\lambda_t = \lambda_\mathrm{m}(t)$ in $[0,T_1)$. The point process T_n, $n \geqslant 1$, defined by model (13)–(14), can be nicely interpreted as a generalized renewal process Kijima (1989), [7]. Indeed, at failure times, as it was mentioned before, λ_t is restored to the predetermined level given by the function $\lambda_\mathrm{P}(t)$.

The following result [9] states that inequality (18) is valid for a more general case:

Proposition 1. Let λ_t, $t\geqslant 0$, be the stochastic intensity defined by the model (15)–(17). Let the values of $\lambda(t,z)$ be ordered with respect to z

$$\lambda(t,z_1)<\lambda(t,z_2),\quad z_1<z_2, \forall z_1,z_2\in[a,b], t\geqslant 0. \tag{19}$$

Then relation (18) holds.

Therefore, the stochastic intensity of the *information-based minimal* repairs λ_t is contained between functions $\lambda_P(t)$ and $\lambda_m(t)$ and equals $\lambda_P(t)$ only at failure points, whereas $\lambda_t=\lambda_m(t)$ in $[0,t_1]$.

The results of this section show that the simplest minimal repair homogeneous model with deterministic stochastic intensity $\lambda_t=\lambda(t)$ in the presence of heterogeneity turns into a model with stochastic intensity described by Proposition 1.

4. A new model of imperfect repair

Firstly, we define this new model for the conventional homogeneous case. Assume, as previously that the failure rate $\lambda(t)$ of the governing distribution is monotonically increasing (IFR). Let the first failure occurs at $t=t_1$. Define the imperfect repair as repair, decreasing the pre-failure value of $\lambda(t)$ in the following way:

$$\lambda(t,t_1,k_1)=\begin{cases}\lambda(t), & 0\leqslant t<t_1,\\ k_1\lambda(t), & t\geqslant t_1,\end{cases} \tag{20}$$

where k_1 is an improvement factor:

$$\frac{\lambda(0)}{\lambda(t_1)-\lambda(0)}\leqslant k_1\leqslant 1.$$

The case $k_1=1$ corresponds to a minimal repair. The subsequent cycles are defined in a similar way, which results in the following stochastic intensity (compare with (3)):

$$\lambda_t=\sum_{n=0}^{\infty}\prod_{i=0}^{n}k_i\lambda(t-T_n)I(T_n\leqslant t<T_{n+1}),$$

which means that on the $n+1$th cycle the failure rate is $k_1k_2\cdots k_n\lambda(t)$. The improvement factors for the following cycles are subject to the similar condition as for the second one:

$$\frac{\lambda(t_i)}{\lambda(t_{i+1})-\lambda(t_i)}\leqslant k_{i+1}\leqslant 1,\quad i\geq 1, t_0\equiv 0.$$

This type of imperfect repair was not studied in the literature so far, but our goal in this paper is not to investigate its properties, which, in fact, could be interesting, but to describe the impact of heterogeneity in this model.

Relation (20) defines a *proportional hazards (PH) model of imperfect repair*. We shall show now that this proportionality is violated in the heterogeneous case. In a general heterogeneous setting (8)–(10), assume a specific multiplicative case of a frailty model:

$$\lambda(t,z)=z\lambda(t) \tag{21}$$

where $\lambda(t)$ is a baseline failure rate. Combining the PH repair model at the time of the first failure (20) with the multiplicative frailty model (21):

$$\lambda(t,t_1,z,k_1)=\begin{cases}z\lambda(t), & 0\leqslant t<t_1,\\ k_1z\lambda(t) & t\geqslant t_1.\end{cases} \tag{22}$$

Thus, in each realization of a frailty Z we have the similar 'proportional repair action' as in the homogeneous case (20). What happens with the corresponding mixture (observed) failure rate? As in (9), the mixture failure rate for the specific model (21) is denoted by $\lambda_m(t)$. Denote the mixture failure rate for the model

$$\lambda(t,z,k_1)=k_1z\lambda(t),\quad t\geqslant 0$$

by $\lambda_{mk_1}(t)$. Then the mixture failure rate for the model (22) is obviously

$$\lambda_m(t,t_1,k_1)=\begin{cases}\lambda_m(t), & 0\leqslant t<t_1,\\ \tilde{\lambda}_{mk_1}(t), & t\geqslant t_1,\end{cases}$$

where $\tilde{\lambda}_{mk_1}(t)$ is a resulting mixture failure rate in $[t_1,\infty)$. Using the same reasoning as in [8], it can be proved that

$$\lambda_{mk_1}(t)>\tilde{\lambda}_{mk_1}(t),\quad \forall t\in[t_1,\infty)$$

and that proportionality is preserved only at $t=t_1$:

$$\tilde{\lambda}_{mk_1}(t_1)=k_1\lambda_m(t_1). \tag{23}$$

Obviously the same considerations are valid for the whole process of repair with repairs at $t_1,t_2,\ldots$.

Thus it is shown that the homogeneous proportional model of repair (20) does not hold in the heterogeneous case, although it holds for each realization of the frailty parameter Z.

The next section will be devoted to describing two models of heterogeneity in the lifesaving framework [11]. The first one will consider probability $\theta(t)$ as a random variable, whereas the second one is based on the doubly stochastic Poisson process.

5. Heterogeneous lifesaving model

In this section, we are coming back to the lifesaving model of Section 2. Let $\theta(t)$ in Eqs. (5) and (6) be constant in time for simplicity: $\theta(t)=\theta$ and assume that it is a random variable (independent of the non-homogeneous process P_t, $t\geqslant 0$) with support in [0,1]. This is another way of implementing the population heterogeneity into the model. Indeed, for instance, probability of survival after some diseases can vary a lot in the population of the same age. The same argument holds for repairable items from different subpopulations (for instance, of different makes).

It follows from [14] that, as relation (7) is valid conditionally on realizations of θ, the following formulas holds, when θ is a random variable:

$$F(t) = 1 - E\left[\exp\left\{-\theta \int_0^t \lambda(u)\,\mathrm{d}u\right\}\right], \tag{24}$$

$$\lambda_{\mathrm{r}}(t) = \lambda(t)E[\theta | T \geqslant t]. \tag{25}$$

Eqs. (24) and (25) define, in fact, the mixture (observed) Cdf and the mixture (observed) failure rate, respectively. The shape of $\lambda_{\mathrm{r}}(t)$ in (25) (e.g., for the arbitrary continuous increasing function) can be already different from the shape of a baseline $\lambda(t)$: it can even decrease for sufficiently large t [10]. Thus the impact of heterogeneity of the described type is in changing the shape of the observed failure rate compared with the baseline one.

Another and maybe much more important source of heterogeneity in harmful events can be modeled by the doubly stochastic Poisson process $\hat{P}_t$, $t \geqslant 0$ (instead of the Poisson process of harmful events). Denote a random rate of this process by $\lambda(t,\Psi)$, where Ψ is a random variable with support in $[0,\infty)$. For the fixed $\Psi = \psi$, similar to relations (6) and (7), we have

$$F(t|\psi) = 1 - \exp\left\{-\int_0^t \theta(u)\lambda(u,\psi)\,\mathrm{d}u\right\}, \tag{26}$$

$$\lambda_r(t) = \theta(t)\lambda(t,\psi) \tag{27}$$

and similar to (24) and (25), we have

$$F(t) = 1 - E\left[\exp\left\{-\int_0^t \theta(u)\lambda(u,\psi)\,\mathrm{d}u\right\}\right], \tag{28}$$

$$\lambda_{\mathrm{r}}(t) = \theta(t)E[\lambda(t,\Psi)|T > t]. \tag{29}$$

There can be different models for $\lambda(t,\Psi)$. The multiplicative one is the simplest one:

$$\lambda(t,\Psi) = \Psi\lambda(t) \tag{30}$$

where $\lambda(t)$, as usually in the proportional hazards-type models, plays the role of a baseline (reference) hazard rate. Then Eq. (29) turns into

$$\lambda_{\mathrm{r}}(t) = \theta(t)\lambda(t)E[\Psi|T > t]. \tag{31}$$

We assume that the baseline function $\lambda(t)$ in Eqs. (30) and (31) is increasing. As previously, the observed failure rate, however, can have a different shape due to the fact that conditional expectations in these formulas are decreasing in time [8]. The following example shows that $\lambda_{\mathrm{r}}(t)$ can even tend to 0 as $t \to \infty$.

Example 2. Consider harmful events with linearly increasing rate

$$\lambda(t,\Psi) = \Psi t$$

and assume that Ψ is gamma distributed with parameters β and ϑ. Then [8]

$$\lambda(t) = \frac{\beta t}{\vartheta + t^2}.$$

This function is equal to zero at $t = 0$ and tends to zero as $t \to \infty$ with a single maximum at $t = \sqrt{\theta}$. Hence, the mixture of IFR distributions has a decreasing (tending to zero!) failure rate for sufficiently large t and this is rather surprising. Furthermore, the same result asymptotically holds for $\lambda(t) = t^{\alpha}$, $\alpha > 0$.

It is worth noting that the doubly stochastic Poisson process effectively models the diversity among subpopulations (and on the 'individual level' as well) in the rate of harmful events. Lifestyle, external factors, hereditary factors etc. are the sources of heterogeneity in biological applications.

6. Conclusions

Repeated *statistical minimal* repair in homogeneous populations gives rise to the non-homogeneous Poisson process of failures (repairs), which is a nice and simple model to deal with. Mixture of distributions is a useful tool for modeling heterogeneity. It is well known that the shape of the failure rate in heterogeneous populations can differ dramatically from the baseline failure rate. On the other hand, it is not trivial to define the analog of *minimal* repair for heterogeneous populations. A version of the *information-based minimal* repair for this case is suggested and analyzed in Section 2.

A new model of imperfect repair, suggested in this paper needs further development. We just showed that the 'transition' from the homogeneous to the heterogeneous case is also non-trivial in this case.

The lifesaving procedure of this paper can be also interpreted in terms of minimal repairs. Different ways of modeling heterogeneity in this model are considered.

We show that heterogeneity is an important factor in reliability analysis, as there are no absolutely homogeneous populations in practice. The diversity among populations can have a dramatic impact on reliability characteristics such as the observed failure rate and should be taken into account in practice.

References

[1] Arjas E, Norros I. Change of life distribution via hazard transformation: an inequality with application to minimal repair. Math Oper Res 1989;14:355–61.
[2] Aven T. Optimal replacement under a minimal repair strategy: a general failure model. Adv Appl Prob 1983;15:198–211.
[3] Aven T, Jensen U. Stochastic models in reliability. Springer; 1999.
[4] Aven T, Jensen U. A general minimal repair model. J Appl Prob 2000;37:187–97.
[5] Block HW, Borges W, Savits TH. Age-dependent minimal repair. J Appl Prob 1985;22:370–86.
[6] Boland PJ, El-Neweihi E. Statistical and information based minimal repair for k out of n systems. J Appl Prob 1998;35:731–40.
[7] Finkelstein MS. Modeling a process of non-ideal repair. In: Limnios N, Nikulin M, editors. Recent Advances in Reliability Theory. Birkhauser; 2000. p. 41–53.
[8] Finkelstein MS, Esaulova V. Modeling a failure rate for a mixture of distribution functions. Prob Eng Inf Sci 2001;15:383–400.
[9] Finkelstein MS. Minimal repair in heterogeneous populations. J Appl Prob 2004;41:281–6.

[10] Finkelstein MS. A model of biological aging and the shape of the observed hazard rate. Lifetime Data Anal 2003;1:93–109.
[11] Finkelstein MS. On some reliability approaches to human aging. Int J Reliab Qual Safety Engineering 2005;12:337–46.
[12] Vaupel JW, Manton KG, Stallard E. The impact of heterogeneity in individual frailty on the dynamics of mortality. Demography 1979;16: 439–54.
[13] Vaupel JW, Yashin AI. Repeated resuscitation: how life saving alters life tables. Demography 1987;4:123–35.
[14] Yashin AI, Manton KG. Effects of unobserved and partially observed covariate processes on system failure: a review of models and estimation strategies. Stat Sci 1997;12:20–34.