ON SOME AGING PROPERTIES OF GENERAL REPAIR PROCESSES

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ABSTRACT. Aging properties of a general repair process are considered. Under certain assumptions it is proved that the expectation of an age at the beginning of the next cycle in this process is smaller than the initial age of the previous cycle. Using this reasoning it is shown that the sequence of random ages at the start (end) of consecutive cycles is stochastically increasing and is converging to a limiting distribution. Possible applications and interpretations are discussed.

Keywords: general renewal process, virtual age, degradation, aging distributions, failure rate, mean remaining lifetime, repairable systems.

1. INTRODUCTION

A convenient mathematical description of repair processes uses a concept of stochastic (or failure) intensity (Aven and Jensen, 1999). Consider, e.g., a renewal process (perfect, instantaneous repair) with an underlying absolutely continuous distribution, $F(t), t \in [0,\infty)$, a failure rate $\lambda(t)$ and a sequence of waiting times $\{S_n\}, n \geq 1, S_0 = 0$. Denote the sequence of i.i.d inter-arrival times by $\{T_n\}, n \geq 1; S_1 = T_1$. The stochastic intensity in this case is compactly written via the corresponding indicator function as

$$\lambda_t = \sum_{n=0}^{\infty} \lambda(t - S_n) I(S_n \le t < S_{n+1}); t \ge 0, \quad (1)$$

Denote by A_t the *age process***,** which corresponds to the renewal process (1):

$$A_t = \sum_{n=0}^{\infty} (t - S_n) I(S_n \le t < S_{n+1}); t \ge 0. \quad (2)$$

Thus, A_t starts at $t = 0$ as a linear function with a unit slope. It jumps down to 0 at S_1, which is the time of the first renewal, etc. The age of a repairable system in this case is just the time elapsed since the last renewal.

As a minimal repair does not change the age of a system, its age process is trivial:

$$A_t = t; t \ge 0. \quad (3)$$

Assume now that a repair action at $t = t_1$ (realization of T_1) decreases the age of a system not to 0 as in the case of a perfect repair, but to $v_1 = qt_1, 0 < q < 1$, and the system starts the second cycle with this initial age in accordance with the Cdf $1 - \bar{F}(v_1 + t) / \bar{F}(v_1)$, where $\bar{F} \equiv 1 - F$. This age is often called the virtual age. For convenience we will omit the term "virtual" in what follows. The forthcoming cycles are defined in a similar way to form a *process of general repair* (Kijima, 1989; Stadje and Zuckerman, 1991; Finkelstein, 1992a, 2000; Baxter *et al.*, 1996; Last and Szekli, 1998, to name a few). The sequence of ages after the ith repair $\{V_i\}_{i \ge 0}$ in this model for a specific case of a linear, deterministic repair function qt is defined as

$$V_0 = 0; V_1 = qT_1; V_2 = q(V_1 + T_2), \ldots, V_i = q(V_{i-1} + T_i), \ldots \quad (4)$$

and distributions of the corresponding inter-arrival times for realizations v_i are given by:

$$\bar{F}_i(t) = \frac{\bar{F}(v_{i-1} + t)}{\bar{F}(v_{i-1})}, i \ge 1. \quad (5)$$

Therefore, the age process for this model is

$$A_t = \sum_{n=0}^{\infty} (t - S_n + V_n) I(S_n \le t < S_{n+1}); t \ge 0. \quad (6)$$

Other settings and generalizations can be also considered (see, e.g., Last and Szekli, 1998 for relevant examples). All these models have a crucial common feature: the corresponding age processes are defined by the generic distribution $F(t)$ and only the 'position' of the starting point of each cycle (as, e.g., in (5)) depends on the concrete model.

Define a stochastic point process as *stochastically aging,* if its inter-arrival times $\{T_n\}, n \geq 1$ are stochastically decreasing:

$$T_{i+1} \leq_{st} T_i, i \geq 1 \ , \tag{7}$$

Thus, the renewal process is not aging in this sense, whereas the non-homogeneous Poisson process is aging, if its rate is an increasing function.

The following definition deals with aging properties of the sequence of ages at the start (end) of cycles for the point processes of the described types.

Definition. *The age process* is called *stochastically increasing,* if the (embedded) sequence of ages at the start (end) of cycles is stochastically increasing.

If, e.g., a generic $F(t)$ is IFR, then the stochastically increasing age process describes overall deterioration of our repairable system with time, which is the case for various wearing out systems in practice.

In what follows we will study the properties of the age process (6) with a non-linear quality of repair function $q(t)$. Under rather weak assumptions it will be shown that this process is stochastically increasing and is becoming stable in distribution (converges to a limiting distribution as $t \to \infty$). These issues for the linear $q(t)$ were first addressed in Finkelstein (1992a), where the corresponding renewal-type equations were also derived. The rigorous and detailed treatment of monotonicity and stability for rather general age processes driven by generic $F(t)$ was given by Last and Szekli (1998). The approach of these authors is based on applying some fundamental probabilistic results: A Loynes-type scheme and Harris-recurrent Markov chains were used. Our approach for a more specific model (but with weaker assumptions on $F(t)$ and with a time dependent $q(t)$) is based on a direct probabilistic reasoning and on the appealing 'geometrical' notion of an equilibrium age v^*.

Apart from obvious engineering applications**,** these results may have some important biological interpretation. Most biological theories of aging agree that the process of aging can be considered as some process of "wear and tear" (see, e.g., Yashin *et al*, 1999). The existence of repair mechanisms in organisms decreasing the accumulated damage on various levels is also a well established fact. As in the case of DNA mutations in the process of cells replication, this repair is not perfect. Asymptotic stability of the repair process means that an organism, as a repairable system, is practically not aging in the defined sense for sufficiently large t. Therefore, the deceleration of human mortality rate at advanced ages (see, e.g., Thatcher, 1999) and even its approaching the mortality plateau can be explained in this way. As it will be seen, this conclusion rely on an important assumption that a repair action decreases the overall accumulated damage and not only its last increment. It is worth noting that another possible explanation of mortality deceleration phenomenon at advanced ages is via the concept of population heterogeneity (see Finkelstein and Esaulova, 2006 for mathematical details).

2. THE QUALITY OF REPAIR FUNCTION

Assume now that a linear function qt in (4) is now an *increasing*, continuous in $[0,\infty)$ and concave function $q(t)$; $q(0)=0$. Therefore:

$$q(t_1+t_2)\le q(t_1)+q(t_2),\ t_1,t_2\in[0,\infty). \tag{8}$$

Let also

$$q(t)<q_0 t\ , \tag{9}$$

where $q_0<1$, which shows that repair rejuvenates the failed item to some extent and that $q(t)$ cannot be arbitrary close to $q(t)=t$ (minimal repair).

Let a cycle start with an age v. Denote by $T(v)$ the cycle duration: The remaining lifetime with a survival function given by the right hand side of equation (5) for $v_{i-1}=v$. The next cycle will start at a *random* age $q(v+T(v))$. In this section we will be interested in some equilibrium age v^*. Define this age as a solution of the following equation:

$$E[q(v+T(v))]=v\,. \tag{10}$$

Thus, if some cycle of a general repair process starts at age v^*, then the next cycle will start with a random age which expectation is v^*, which is obviously a martingale property.

Theorem 1. *Let* $\{T_n\}, n \geq 1$ *be a process of general repair with an increasing, continuous quality of repair function* $q(t)$*, defined by relations (8) and (9).*

Assume that the generic distribution $F(t)$ *has a finite first moment and that the corresponding failure rate is either bounded from below for the sufficiently large* t *by* $c > 0$ *or is converging to* 0 *as* $t \to \infty$ *such that*

$$\lim_{t \to \infty} t\lambda(t) = \infty . \tag{11}$$

Then there exists, at least, one solution of equation (10), and if there are more than one, the set of these solutions is bounded in $[0, \infty)$.

Proof. a. As $E[T(0)] < \infty$, it is evident, that $E[T(v)] < \infty, v > 0$. If $\lambda(t)$ is bounded from below by $c > 0$, then

$$E[T(v)] \leq \frac{1}{c},$$

Therefore, applying (8)

$$E[q(v + T(v)] \leq q(v) + E[T(v)] . \tag{12}$$

It follows from (9) and (12)) that

$$E[q(v + T(v))] < v$$

for sufficiently large v. On the other hand, $E[q(T(0))] > 0$, which proves the first part of this theorem, as the function $E[q(v + T(v))] - v$ is continuous in v, positive at $v = 0$ and negative for sufficiently large v.

b. Let now $\lambda(t) \to 0$ as $t \to \infty$. Consider the following quotient:

$$\frac{E[T(v)]}{v}=\frac{\int_v^{\infty}\exp\left\{-\int_0^x \lambda(u)du\right\}dx}{v\exp\left\{-\int_0^v \lambda(u)du\right\}}.$$

Applying the L'Hopital's rule and using assumption (11)

$$\lim_{v\to\infty}\frac{E[T(v)]}{v}=\lim_{t\to\infty}\frac{1}{\lambda(v)v-1}=0. \tag{13}$$

Therefore, applying inequality (12) and taking into account relations (9) and (13),

$$\frac{E[q(v+T(v))]}{v}\le\frac{q(v)}{v}+\frac{E[T(v)]}{v}<1.$$

The last inequality holds for sufficiently large v. Using the same argument as in the first part of the proof, completes our reasoning.

Corollary. If $F(t)\in IFR$, then the conditions of Theorem 1 hold and there is, at least, one solution of equation (10).

Remark 1. Sufficient condition (11) is a rather weak one stating, in fact, that $t\lambda(t)$ must just have a limit as $t\to\infty$, which should not be finite. For instance, the 'bizarre' failure rate $\lambda(t)=\frac{|\sin t|\ln t}{t}$, which tends to 0 as $t\to\infty$, does not comply with (11). On the other hand, it is clear that for the, e.g., Weibull distribution with the decreasing failure rate relation (11) holds.

Theorem 2. *Let* $F(t)\in$ *IFR. Assume that a current cycle of a general repair process start at age* $v^*+\Delta v$*, where* v^* *is an equilibrium solution of equation (10) and* $\Delta v>0$.

Then the expectation of an age at the start of the next cycle 'will be closer' to v^**:*

$$v^*<E[q(v^*+\Delta v+T(v^*+\Delta v))]<v^*+\Delta v. \tag{14}$$

Proof. As stated in the corollary to Theorem 1, at least, one solution of equation (10) exists in this case Let us prove first the right inequality in (14). Taking into account that

$q(t)$ is an increasing function and that random variables $T(v)$ are stochastically decreasing in v (for increasing $\lambda(t)$):

$$E[q(v^*+\Delta v+T(v^*+\Delta v))] < E[q(v^*+\Delta v+T(v^*))]. \qquad (15)$$

When obtaining this inequality the following simple fact was used: If two distributions are ordered as $\overline{F}_1(t) > \overline{F}_2(t), t \in (0,\infty)$ and $g(t)$ is an increasing function, then integrating by parts it is easy to see that

$$\int_0^\infty g(t)dF_2(t) < \int_0^\infty g(t)dF_1(t). \qquad (16)$$

Finally:

$$E[q(v^*+\Delta v+T(v^*))] \le E[q(v^*+T(v^*))] + q(\Delta v) = v^* + q(\Delta v) < v^*+\Delta v.$$

The left inequality in (14) is proved using the similar arguments:

$$E[q(v^*+\Delta v+T(v^*+\Delta v))] > E[q(v^*+T(v^*))] = v^*$$

after observing that the random variable $v^*+\Delta v+T(v^*+\Delta v)$ is stochastically larger than $v^*+T(v^*)$.

Corollary. *If $F(x) \in IFR$, equation (10) has a unique solution.*

Proof. Assume that there are two solutions of equation (10):

$$E[q(v^*+T(v^*))] = v^*, \qquad (17)$$

$$E[q(\tilde{v}+T(\tilde{v})] = \tilde{v}. \qquad (18)$$

Let $\tilde{v} = v^*+\Delta v, \Delta v > 0$. Then, in accordance with (14):

$$E[q(\tilde{v}+T(\tilde{v}))] = E[q(v^*+\Delta v+T(v^*+\Delta v))] < v^*+\Delta v = \tilde{v},$$

which contradicts (18).

Remark 2. When equation (10) has a unique solution, it can be shown similar to (14), that

$$v^*-\Delta v < E[q(v^*-\Delta v+T(v^*-\Delta v))] < v^*.$$

Remark 3. The results of this section hold when repair action is stochastic: $\{Q_i\}, i \geq 1$ is a sequence of i.i.d random variables (independent from other stochastic components of the model) with support in $[0,1]$ and $E[Q_i] < 1$.

The described properties show that there is a shift in the direction of the equilibrium point v^* of the starting age of the next cycle as compared with the starting age of the current cycle. Note that for the minimal repair process the corresponding shift is always in the direction of infinity.

3. MONOTONICITY AND STABILITY OF AN AGE PROCESS

Denote the age distribution at the start of the $(i+1)$th cycle by $\theta_{i+1}^S(v)$, $i = 1,2,....$ and by $\theta_i^E(v), i = 1,2,...$ the corresponding age distribution at the end of the previous i th cycle. It is clear that in accordance with our model (4)-(5), (8)-(9):

$$\theta_{i+1}^S(v) = \theta_i^E(q^{-1}(v)), i = 1,2,... \tag{19}$$

where the inverse function $q^{-1}(v)$ is also increasing. This can be easily seen, as

$$\theta_{i+1}^S(v) = \Pr(V_{i+1}^S \leq v) = \Pr(q(V_i^E) \leq v) = \Pr(V_i^E \leq q^{-1}(v)),$$

where V_{i+1}^S and V_i^E are random ages at the start of $(i+1)$ th cycle and at the end of the previous one, respectively The following theorem states that the age processes under consideration are stochastically increasing.

Theorem 3. *Random ages at the end (start) of each cycle in the general repair model (4)-(5) and (8)-(9) form the stochastically increasing sequences:*

$$\bar{\theta}_{i+1}^E(v) > \bar{\theta}_i^E(v), (\bar{\theta}_{i+2}^S(v) > \bar{\theta}_{i+1}^S(v)), v > 0, i = 1,2,.... \tag{20}$$

Proof. We shall prove the first inequality; the second one trivially follows from (19). Consider the first two cycles. Let v_1^E be the realization of T_1-the age at the end of the first cycle and at the same time the duration of this cycle. Then (for this realization) the age at the end of the second cycle is $q(v_1^E)+T(q(v_1^E)$. It is clear that this random variable is stochastically larger than T_1, and as this property holds for each realization, inequalities (20) hold for $i=1$.

Assume that inequalities (20) hold for $i=n-1, n\geq 3$. Due to definition of age at the start and the end of a cycle, integrating by parts and using relation (19):

$$\begin{aligned}\theta_n^E(v) &= \int_0^v \left(1-\exp\left\{-\int_x^v \lambda(u)du\right\}\right) d\left[\theta_n^S(x)\right] = \int_0^v \theta_n^S(x) d_x\left(\exp\left\{-\int_x^v \lambda(u)du\right\}\right) \\ &= \int_0^v \theta_{n-1}^E(q^{-1}(x)) d_x\left(\exp\left\{-\int_x^v \lambda(u)du\right\}\right),\end{aligned} \qquad (21)$$

$$\begin{aligned}\theta_{n+1}^E(v) &= \int_0^v \left(1-\exp\left\{-\int_x^v \lambda(u)du\right\}\right) d\left[\theta_{n+1}^S(x)\right] = \int_0^v \theta_{n+1}^S(x) d_x\left(\exp\left\{-\int_x^v \lambda(u)du\right\}\right) \\ &= \int_0^v \theta_n^E(q^{-1}(x)) d_x\left(\exp\left\{-\int_x^v \lambda(u)du\right\}\right),\end{aligned} \qquad (22)$$

where the fact that $\exp\left\{-\int_x^{x+(v-x)} \lambda(u)du\right\} = \exp\left\{-\int_x^v \lambda(u)du\right\}$ is the probability of survival from initial age x to age $v>x$ was used. This can also be interpreted via the remaining lifetime concept.

Taking into account the induction assumption and comparing (21) and (22), using similar reasoning, as while obtaining inequality (16), results in

$$\theta_n^E(v) < \theta_{n-1}^E(v) \Rightarrow \theta_n^E(q^{-1}(v)) < \theta_{n-1}^E(q^{-1}(v)) \Rightarrow \theta_{n+1}^E(v) < \theta_n^E(v)$$

and this completes the proof.

The next theorem states that the increasing (decreasing) sequences of survival (distribution) functions $\bar{\theta}_i^E(v), \bar{\theta}_i^S(v)(1-\bar{\theta}_i^E(v), 1-\bar{\theta}_i^S(v))$ are converging to a limiting survival (distribution) function as $i\to\infty$. Thus the repair process is stable in the defined sense.

Theorem 4. *Let the governing distribution* $F(t)$ *in a general repair model (4)-(5), (8)-(9) be IFR.*

Then there exist the limiting distributions for ages at the start and at the end of cycles:

$$\lim_{i\to\infty}\theta_i^E(v)=\theta_L^E(v),(\lim_{i\to\infty}\theta_i^S(v)=\theta_L^S(v)). \tag{23}$$

Proof. The proof is based on theorems 2 and 3. As the sequences (20) are increasing at each $v>0$, there can be only 2 possibilities: Either there are limiting distributions (23) with uniform convergence in $[0,\infty)$, or the ages grow infinitely, as for the case of minimal repair: $q=1$. The latter means that for each fixed $v>0$:

$$\lim_{i\to\infty}\theta_i^E(v)=0,(\lim_{i\to\infty}\theta_i^S(v)=0). \tag{24}$$

Assume that (24) holds and consider the sequence of ages at the start of a cycle. Then for an arbitrary small $\varsigma>0$ we can find n such that

$$\Pr(V_i^S\le v^*)\le\varsigma, i\ge n, \tag{25}$$

where v^* is an equilibrium point, which is unique and finite according to the corollary to Theorem 2. It follows from inequalities (14) that for each realization $v_i^s>v^*$ the expectation of the starting age at the next cycle is smaller than v_i^s. On the other hand, the 'contribution' of ages in $[0,v^*)$ can be made arbitrary small, if (24) is true. Therefore, it can be easily seen that for the sufficiently large i:

$$E[V_{i+1}^S]<E[V_i^S].$$

This inequality contradicts Theorem 3, according to which expectations of ages form an increasing sequence. Therefore assumption (24) is wrong, and the limiting property (23) holds. As previously, the result for the second limit in (23) trivially follows from (19).

Corollary. *The sequence of inter-arrival lifetimes* $\{T_n\},n\ge1$ *is stochastically decreasing to a random variable with a limiting distribution:*

$$\lim_{i\to\infty} F_i(t) = F_L(t) = \int_0^\infty \left(1-\exp\left\{-\int_v^{v+t}\lambda(u)du\right\}\right)d(\theta_L^S(v)). \qquad (26)$$

Proof. Equation (26) follows immediately after taking into account that convergence in (23) is uniform. On the other hand, comparing:

$$F_i(t) = \int_0^\infty \left(1-\exp\left\{-\int_v^{v+t}\lambda(u)du\right\}\right)d(\theta_i^S(v))$$

and

$$F_{i+1}(t) = \int_0^\infty \left(1-\exp\left\{-\int_v^{v+t}\lambda(u)du\right\}\right)d(\theta_{i+1}^S(v))$$

it is easy to see, using the same argument as when proving Theorem 2, that $F_{i+1}(t) > F_i(t), t > 0; i = 1,2,...$, (stochastically decreasing sequence of inter-arrival times), as $\theta_{i+1}(v) < \theta_i(v)$ and the integrand function is increasing in v for the IFR case.

CONCLUDING REMARKS

Under reasonable assumptions we show that the general repair process with a quality of repair function $q(t)$, defined by relations (8) and (9), is stochastically increasing. Therefore, this property describes a certain overall deterioration of a repairable object. On the other hand, Theorem 4 states that this deterioration slows down and eventually vanishes at the infinity, which means that the defined type of repair is decreasing age (wear) in a 'sufficient' for this result way.

Model (4) is usually referred to in the literature as Kijimaa-2 general repair model. It is worth noting that it was independently suggested and analyzed in Finkelstein (1988). Unfortunately this paper was not translated into English..

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