Linking Period and Cohort Life Expectancy in Gompertz Proportional Hazards Models

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Abstract

Adult mortality decline was the driving force of life-expectancy increase in many developed countries in the second half of the twentieth century. In this paper we study one of the most widely used models to capture adult human mortality – the Gompertz proportional hazards model. In its standard settings we, first, derive analytic expressions for period and cohort life expectancy. In addition we formulate a necessary and sufficient condition for the unboundedness of life expectancy. Secondly, we prove that if mortality decreases in time at all ages by the same proportion, both period and cohort life expectancy at birth increase linearly. Finally, we derive simple formulae that link period and cohort life expectancy to one another. They imply that if period life expectancy at birth increases steadily by three months per year, which has been the case for the best-practice country since 1840, then the corresponding cohort life expectancy rises constantly by four months per year.

Introduction

Mortality in the developed countries decreased steadily in the second half of the twentieth century (Tuljapurkar, Li and Boe 2000). This lead to an increase in human life expectancy at birth by more than 15%, from 71.28 years in 1945 to 82.87 years in 2007, for females in the best-practice country (HMD 2010). Moreover, the observed pace was almost constant (Oeppen and Vaupel 2002; White 2002). On the other hand, observing the shifts in the survival function of human populations over time, Bongaarts and Feeney (2003) introduced the idea of tempo effects in mortality. Goldstein (2008) pointed out that Bongaarts and Feeney focused on tempo adjustments of period life expectancy, while neglected the implications of tempo effects on cohort life expectancy. Goldstein also shows that, assuming linearly shifting mortality, the tempo-adjusted period life expectancy is the same as the life expectancy of the cohort born as many years earlier as the tempo-adjusted period life expectancy itself is equal to. Working in Goldstein's model settings, we restrict ourselves, on the one hand, by studying a Gompertz proportional hazards model. On the other hand, we take one step further as we derive a relationship between increases in period and cohort life expectancy at birth.

It has been known since Gompertz (1825) that humans at adult ages experience an exponentially increasing force of mortality. The driving force of life expectancy increase since the 1960s has been the mortality reduction of ages over 45 (Cutler and Meara 2004). Kannisto (1994) argues that old-age mortality decline could capture period effects of mortality improvement. Moreover, as Bongaarts and Feeney (2003) noted, proportionally changing adult mortality serves as a reasonable model to account for such amelioration in countries with high life expectancy.[1] A Gompertz force of mortality that changes in time at all ages by an equal proportion represents a model that captures the mortality schedule observed by Kannisto and Bongaarts. The meaning of this model is that proportional mortality reductions keep young-age mortality relatively stable while influencing middle- and old-age mortality.

[1]For criticisms of the proportionality assumption please refer to e.g. Guillot (2008) or Wachter (2008).

In this article we study a Gompertz proportional hazards model for a homogeneous population with a fixed shape and a monotonically decreasing scale parameter. We prove that in these settings (i) the age distribution of life-expectancy years gained is linearly shifting, (ii) this linear shift equals the change in period life expectancy at birth, (iii) the assumption about proportionally declining force of mortality implies that life expectancy at birth increases infinitely over time. In addition, based on the linearly shifting age distribution of the population, we derive simple relationships linking period and cohort life expectancy at birth, as well as their time derivatives.

The Unlimited Human Life Expectancy

First of all, we would like to prove a rather intuitive statement. Namely we will show that in a Gompertz proportional hazards model with a fixed shape parameter life expectancy grows infinitely, given there is steady mortality reduction at all ages. In other words we keep the rate of aging constant and assume mortality improvement to be captured just by proportional reduction of the force of mortality. Formally, suppose the force of mortality $\mu(x, y)$ for an individual aged x in year y is given by

$$\mu(x, y) = a(y)e^{bx}\,, \qquad a(y) > 0,\ b > 0,\ x \geq 0,\ y \geq 0 \tag{1}$$

i.e. by a Gompertz curve with a constant shape parameter $b > 0$ and a time-varying scale parameter $a(y) > 0$. Suppose also that $\lim_{y\to\infty} a(y) = 0$. This is justified for human populations as $a(y)$, estimated from data in HMD (2010), is $O(10^{-6})$.

The number of years lived in the next n by those who have attained age x at time y will be (Keyfitz and Caswell 2005: 30):

$$L(x, n, y) = \int_{x}^{x+n} \ell(v, y)dv \tag{2}$$

where

$$\ell(x,y) = \exp\left\{ -\int_0^x \mu(v,y)dv \right\} \tag{3}$$

is the survival function of an individual aged x at time y.

Substituting (1) and (3) in (2) and assuming $\ell(0,y) = 1$, we get

$$L(x,n,y) = \frac{1}{b}\, e^{\frac{a(y)}{b}} \int_{\frac{a(y)}{b}\, e^{bx}}^{\frac{a(y)}{b}\, e^{b(x+n)}} \frac{e^{-q}}{q} dq \tag{4}$$

By setting $x = 0$ and letting $n \to +\infty$, we can use (4) to get an expression for life expectancy at birth $e_0(y)$:

$$e_0(y) = \frac{1}{b}\, e^{\frac{a(y)}{b}} E_1\left(\frac{a(y)}{b}\right), \tag{5}$$

where $E_1(\cdot)$ is the exponential integral (see, for example, Abramowitz and Stegun 1964: 229). As

$$\lim_{a(y)\to 0+} E_1\left(\frac{a(y)}{b}\right) = +\infty, \tag{6}$$

life expectancy at birth does not have a finite upper limit, i.e.

$$\lim_{a(y)\to 0+} e_0(y) = \lim_{a(y)\to 0+} \frac{1}{b}\, e^{\frac{a(y)}{b}} E_1\left(\frac{a(y)}{b}\right) = +\infty \tag{7}$$

This result is intuitive as $a(y) \to 0+$ means essentially that age-specific mortality rates become negligibly small with time. Nevertheless, (7) provides mathematical evidence that decreasing age-specific mortality rates cause an infinite increase of human life spans and do not lead to mortality compression at a hypothetical maximal age ω.

Formula (5) provides one additional interesting result. Namely that life expectancy can grow infinitely if and only if $a(y) \to 0+$ in our problem settings, i.e. Gompertz proportional hazards model with $b = const$. The necessity of this condition has already been proved. Its sufficiency follows directly from the well-known inequality (Abramowitz and Stegun 1964: 229)

$$\frac{1}{2}\ln\left(1+\frac{2}{q}\right) < e^q E_1(q) < \ln\left(1+\frac{1}{q}\right), \qquad q > 0\,, \tag{8}$$

which provides the upper and lower boundary for $e_0(y)$. Life expectancy at birth is b times the middle term in (8) for $q = \frac{a(y)}{b}$, and is infinite if and only if $q \to 0+$, which is equivalent to $a(y) \to 0+$ for $b = const$. Apparently when $a(y) \to +\infty$, we will have $e_0(y) \to 0$, and if $a(y) \to k$, where $k = const$, then $\frac{1}{2b}\ln\left(1+\frac{2b}{k}\right) < e_0(y) < \frac{1}{b}\ln\left(1+\frac{b}{k}\right)$. The latter means that if mortality rates stay constant at their current level, i.e. for $a(y) = 10^{-6}$ and $b = 0.14$, then according to (8) life expectancy will be between $\frac{1}{2b}\ln\left(1+\frac{0.28}{0.000001}\right) \approx 44.8$ years and $\frac{1}{b}\ln\left(1+\frac{b}{k}\right) \approx 98.7$ years. Although inequality (8) provides rather wide bounds for life expectancy at birth, it nevertheless states that if current mortality conditions stay constant in time, life expectancy at birth cannot exceed 98.7 years.

A Model Describing Constant Increase in $e_0(y)$

Period life expectancy at birth for the best-practice country increased almost linearly from 1840 onwards at a pace of approximately 2.5 years per decade (Oeppen and Vaupel 2002). In this section we would like to determine the class of Gompertz proportional hazards models that produce this constant rate of life-expectancy increase. So far we proved that $a(y) \to 0+$ is a necessary and sufficient condition for unlimited $e_0(y)$. However, if we want to have a uniform increase in life expectancy at birth, we need to specify the speed of convergence of $a(y)$ to 0. It turns out that if mortality progress over time is described by equal proportional changes at all ages, life expectancy at birth increases linearly over time. This class of models provides one of the possible explanations for the observed record life-expectancy curve (Oeppen and Vaupel 2002). The finding is formulated in the following

Theorem 1. *Suppose*

$$\mu(x,y) = a(y)e^{bx}\,, \qquad a(y) > 0,\ b > 0, x \geq 0, y \geq 0$$

and

$$a(y) = (1-r)^y a_0\,, \qquad a_0 > 0,\ 0 \leq r < 1 \tag{9}$$

Then, as $y \to +\infty$

$$\frac{de_0(y)}{dy} = -\frac{\ln(1-r)}{b} \equiv const \tag{10}$$

Proof. First of all, (9) implies

$$\frac{da(y)}{dy} = (1-r)^y \ln(1-r) a_0 \tag{11}$$

Secondly,

$$\frac{de_0(y)}{dy} = \int\limits_0^\infty \frac{dL(x,n,y)}{dy}\,dx = -\frac{1}{b}\frac{da(y)}{dy}\int\limits_0^\infty (e^{bx}-1)\left(e^{-\frac{a(y)}{b}\,(e^{bx}-1)}\right)dx \tag{12}$$

Substituting (11) in (12) results in

$$\frac{de_0(y)}{dy} = -\frac{a_0}{b}(1-r)^y \ln(1-r)\left(\int_0^\infty e^{bx} e^{-\frac{a(y)}{b}\,(e^{bx}-1)} dx - \int_0^\infty e^{-\frac{a(y)}{b}\,(e^{bx}-1)} dx\right)$$

Then, we substitute (9) and introduce a new variable for $e^{-\frac{(1-r)^y a_0}{b}\,(e^{bx}-1)}$ in the first integral, as well as a new variable for $\frac{a(y)}{b}\,e^{bx}$ in the second integral to get

$$\frac{de_0(y)}{dy} = -\frac{\ln(1-r)}{b} + \frac{a_0 \ln(1-r)}{b^2}(1-r)^y e^{\frac{a_0}{b}(1-r)^y} E_1\left(\frac{a_0}{b}(1-r)^y\right)$$

We use further the well-known representation of the exponential integral (see e.g., Abramowitz and Stegun 1964: 229) for real positive arguments:

$$E_1\left(\frac{a_0}{b}(1-r)^y\right) = - \ln\left(\frac{a_0}{b}(1-r)^y\right) - \gamma - \tag{13}$$

$$- \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\left(\frac{a_0}{b}(1-r)^y\right)^n}{n!} \tag{14}$$

where $\gamma \approx 0.57721567$ is the Euler-Mascheroni constant.

Note that $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\left(\frac{a_0}{b}(1-r)^y\right)^n}{n!}$ represents alternating series. As $0 \leq r < 1$, the general term of the series $\frac{\left(\frac{a_0}{b}(1-r)^y\right)^n}{n\,n!}$ is monotone decreasing and approaches zero as $n \to +\infty$. Thus the alternating series are convergent according to the Leibniz' criterion. Let γ^* denote the sum of the alternating series and the Euler-Mascheroni constant. Then we will have

$$\begin{aligned}\frac{de_0(y)}{dy} &= -\frac{\ln(1-r)}{b} + \\ &+ \frac{a_0 \ln(1-r)}{b^2}(1-r)^y e^{\frac{a_0}{b}(1-r)^y}\left(-\ln\left(\frac{a_0}{b}(1-r)^y\right) - \gamma^*\right)\end{aligned}$$

Taking into account

$$\lim_{y\to\infty} \frac{a_0 \ln(1-r)}{b^2}(1-r)^y e^{\frac{a_0}{b}(1-r)^y}\gamma^* = 0$$

and

$$\lim_{y\to\infty} \frac{a_0 \ln(1-r)}{b^2}\,(1-r)^y e^{\frac{a_0}{b}(1-r)^y} \ln\left(\frac{a_0}{b}(1-r)^y\right) = 0\,,$$

we get

$$\lim_{y\to\infty} \frac{de_0(y)}{dy} = -\frac{\ln(1-r)}{b} \qquad \text{Q.E.D.}$$

□

This result was first introduced by Vaupel (1986: 153), and later derived in a slightly different manner by Goldstein and Wachter (2006: 266). Both findings pertained to a Gompertz proportional hazards model with a slightly different parametrization: $\rho = -\ln(1-r)$ (Vaupel 1986), $k = -\ln(1-r)$ (Goldstein and Wachter 2006). While they proved that life expectancy increases *approximately* at a constant rate $-\ln(1-r)/b$ as $y \to +\infty$, Theorem 1 indicates that this asymptotic pace is in fact *exact*.

Age of Maximal Number of e_0 Years Gained

Theorem 1 determines the rate of life-expectancy change for $y \to +\infty$. However, it does not specify the speed of convergence to $-\ln(1-r)/b$, i.e. we do not know when exactly life expectancy's rate of increase will become constant. What the proof of Theorem 1 contains, though, is the rate of change $\frac{de_0}{dy} =: C_y$ at every time instant y:

$$C_y = -\frac{a_0}{b}(1-r)^y \ln(1-r) \int_0^\infty (e^{bx}-1) e^{-\frac{a_0}{b}(1-r)^y (e^{bx}-1)} dx \tag{15}$$

Although we specify just the time index y, C_y depends on age x, too. Fixing y and considering C_y a function of x only, we can construct an age-distribution of the number of life-expectancy years gained in year y. We discuss the changes in this distribution over time in the next section. Here we focus first on the age x^* at which the maximal gains occur and study its shift over time. We will use (15) to find an explicit expression for x^*.

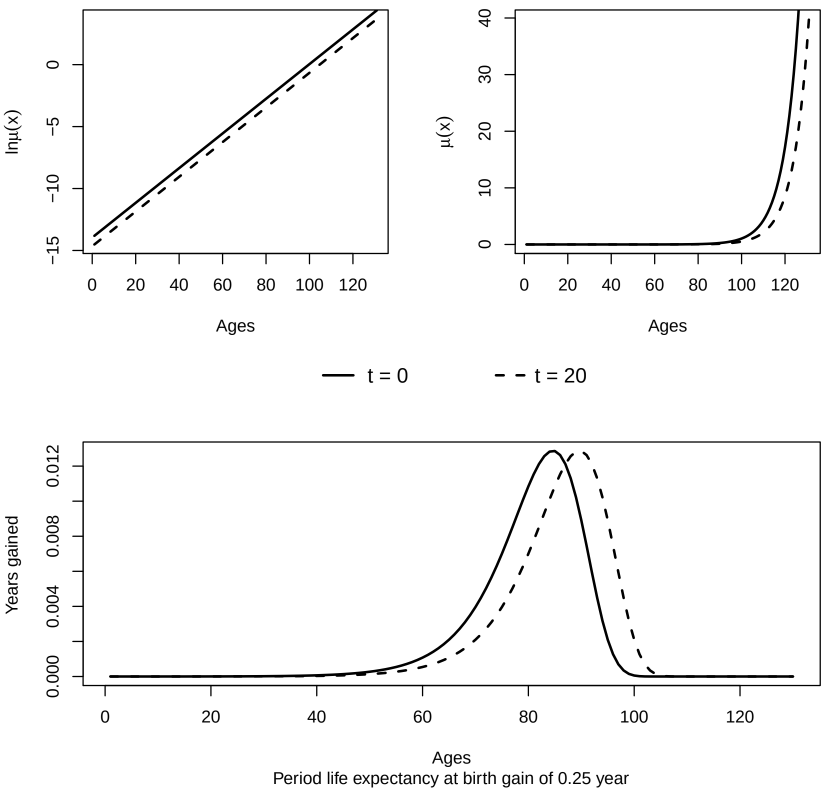

Figure 1: Change in the force of mortality and distribution by ages of years gained by constant period life expectancy at birth increase

While $-\frac{a_0}{b}(1-r)^y \ln(1-r)$ is essentially a scaling factor, the shape of C_y (see Fig. 1) is determined by the behavior of the integrand in (15), which we will denote by $\psi(x,y)$. Namely

$$\psi(x,y) \quad = \quad (e^{bx}-1)e^{-\frac{a_0}{b}(1-r)^y(e^{bx}-1)}$$

The maximum number of person-years gained in an infinitesimal age interval equals the maximum of the function ψ with respect to its first argument. Let $\psi(x,y)$ take its maximal value with respect to x, as already denoted, at x^*, i.e. where the value of its first partial derivative with respect to x is zero and the respective second partial derivative is negative. Then

$$x^* \quad = \quad \frac{\ln(b-(1-r)^y a_0) - y\ln(1-r) - \ln a_0}{b} \tag{16}$$

Note that

$$\frac{dx^*}{dy} \quad = \quad -\frac{\ln(1-r)}{(b-a_0(1-r)^y)}$$

Since $(1-r)^y a_0 \to 0$ as $y \to +\infty$, we have

$$\frac{dx^*}{dy} \quad = \quad -\frac{\ln(1-r)}{b} = \frac{de_0(y)}{dy} \qquad y \to +\infty$$

i.e. the shift of x^* by periods (see Fig. 1) equals the rate of change in life expectancy at birth. Note, however, that x^* is higher (i.e. designates a later age) than the life expectancy at birth of the respective period for $y \to +\infty$. This means that the individuals from the synthetic cohort, corresponding to the life table at time y, do not live on average up to the age at which the biggest gains in $e_0(y)$ take place. This result is given in the following

Theorem 2. *As $y \to +\infty$, in the premises of Theorem 1 and for $\gamma^* > 0$, life expectancy at birth $e_0(y)$ does not exceed x^*, where x^* is the age at which instantaneous increase in $e_0(y)$ is maximal with respect to x, and γ^* is the sum of the Euler-Mascheroni constant and the convergent alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{\left(\frac{a_0}{b}(1-r)^y\right)^n}{n!}$.*

Proof. Using (5) and (13), we can express life expectancy at birth in the following manner:

$$e_0(y) = \frac{1}{b} e^{\frac{a(y)}{b}} \left(-\ln\left(\frac{a_0}{b}(1-r)^y\right) - \gamma^* \right)$$

As $a(y) = a_0(1-r)^y \to 0$ for $y \to +\infty$, we have

$$e_0(y) = \frac{1}{b} \left(-\ln\left(\frac{a_0}{b}(1-r)^y\right) - \gamma^* \right) \qquad y \to +\infty$$

Using (16), we can easily see that

$$x^* = \frac{1}{b} (\ln b - y\ln(1-r) - \ln a_0) \qquad y \to +\infty \tag{17}$$

Finally, when $\gamma^* > 0$

$$\ln b - y\ln(1-r) - \ln a_0 \quad > \quad -\ln\left(\frac{a_0}{b}(1-r)^y\right) - \gamma^*$$

Q.E.D.

□

Note that the number of years gained at x^*, denoted by y^*, becomes also constant as $y \to +\infty$. Indeed, substituting (17) in (15), we get

$$y^* = -\left(\ln(1-r) + \frac{a_0}{b}(1-r)^y \ln(1-r)\right) e^{(1-r)^y \frac{a_0}{b} - 1}$$

which implies

$$y^* = \frac{-\ln(1-r)}{e} \qquad y \to +\infty$$

In addition, as the force of mortality on a logarithmic scale is equal to

$$\ln \mu(x) = t \ln(1-r) + \ln a + bx\,,$$

its rate of change over time $\ln(1-r)$ is one and the same for all ages

$$\frac{d \ln \mu(x)}{dy} = \ln(1-r).$$

As $\ln(1-r) < 0$, the force of mortality decreases at all ages by the same proportion. This phenomenon is most evident at ages where the force of mortality has high values. Nevertheless, in practice, ages at which life-expectancy gains take place are just the ages to which individuals survive (see Fig. 1).[2]

[2]Bongaarts (2009) showed that between 1960 and 2000 in Sweden the logarithm of the age-specific death rates has decreased very similarly to the one in Fig. 1.

Linear Shifts in the Age-Distribution of e_0-years Gained

The Gompertz proportional hazards model with $a(y) = (1-r)^y a_0$ belongs to the Linear Shift Model family (see, for example, Goldstein and Wachter 2006), in which the force of mortality at every age x is given by the force of mortality at a younger age $x - ry$, y years before. Therefore, the age-distribution of life-expectancy years gained will have the same shape with a shifting in time to the older ages mode. We will illustrate this phenomenon by studying (15).

Life expectancy at age x from time y to time τ, $\tau > y$, increases if at time τ there is an age interval $[\xi, \xi + \nu]$, $\xi \geq x$, $\nu > 0$, in which $L(\xi, \nu, \tau) > L(x, \nu, y)$. If the age distribution of life-expectancy years gained retains its shape and shifts its mode to higher ages, then period life expectancy at birth itself rises at a constant pace. Remaining life expectancy at a given age could rise or remain unmodified depending on its position with respect to the interval, in which person-years gains take place. A natural corollary of this phenomenon is that cohort life expectancy can increase by a bigger number of years if mortality reductions take place at higher ages.

In this section we show that the age-distribution of person-years gained shifts linearly over time y. A linear shift implies that the age-derivative of C_y at age x and time y equals the same derivative at age $x + n\rho$ and time $y + n$. Formula (15) implies

$$\frac{dC_y}{dx} = a_0 e^{bx} (1-r)^y \ln(1-r) \left(\frac{a_0}{b} (1-r)^y (e^{bx} - 1) - 1 \right) e^{-\frac{a_0}{b} (1-r)^y (e^{bx} - 1)}$$

Suppose, for the sake of simplicity, that time y is given in discrete units. Then a shift by ρ for a unit change in y implies

$$
\begin{aligned}
\frac{dC_{y+n}}{dx} &= a_0 e^{b(x+n\rho)}(1-r)^{y+n}\ln(1-r)\times \\
&\times\left(\frac{a_0}{b}(1-r)^{y+n}(e^{b(x+n\rho)}-1)-1\right)e^{-\frac{a_0}{b}(1-r)^{y+n}\left(e^{b(x+n\rho)}-1\right)}
\end{aligned}
$$

If for large x we assume

$$
e^{bx}-1\approx e^{bx}\,, \tag{18}
$$

then

$$
\frac{dC_y}{dx}=a_0 e^{bx}(1-r)^y\ln(1-r)\left(\frac{a_0}{b}(1-r)^y e^{bx}-1\right)e^{-\frac{a_0}{b}(1-r)^y e^{bx}}
$$

and

$$
\begin{aligned}
\frac{dC_{y+n}}{dx} &= a_0 e^{b(x+n\rho)}(1-r)^{y+n}\ln(1-r)\left(\frac{a_0}{b}(1-r)^{y+n}e^{b(x+n\rho)}-1\right)\times \\
&\times e^{-\frac{a_0}{b}(1-r)^{y+n}e^{b(x+n\rho)}}
\end{aligned}
$$

From (16) we know that the age at maximal life-expectancy gains x^* is shifted by $-\frac{\ln(1-r)}{b}$ per unit time. Setting $\rho=-\frac{\ln(1-r)}{b}$, we get

$$
\frac{dC_{y+n}}{dx} = A_0 e^{bx}(1-r)^y\ln(1-r)\left(\frac{a_0}{b}(1-r)^y e^{bx}-1\right)e^{-\frac{a_0}{b}(1-r)^y e^{bx}}
$$

Thus, $\frac{dC_y}{dx}=\frac{dC_{y+n}}{dx}$. As most changes in person-years lived occur at later ages (see Fig. 1), the approximation for $\frac{dC_{y+n}}{dx}$ above sufficiently describes the movement of the curve in time.

The area under the curve is constant for all y. Hence the number of years of life expectancy gained at all ages stays the same and only the distribution's mode is shifted by $-y\,\frac{\ln(1-r)}{b}$ (see Fig. 1). Approximation (18) is not restrictive for human populations as the b parameter is approximately 0.11.

The Cohort Life Expectancy at Birth

While studying life expectancy at birth, we have not specified whether we talk about the period or the cohort measure so far. That is merely because the formulae we derived could be easily adjusted from one of the cases to the other by selecting the correct year of birth either of the real (for cohort e_0) or the 'synthetic' cohort (for period e_0). In this section we link these two measures by setting up a relationship that allows calculating cohort life expectancy by knowing the respective period life expectancy and its time-derivative. In addition, we derive a simple formula that expresses the rate of change of cohort life expectancy over time through the respective period life expectancy changes.

The life expectancy at birth of a cohort born at time y depends on the mortality progress in the periods the cohort lives through. As the force of mortality decreases over time, the person-years lived in a certain age interval in a certain time period correspond to the person-years lived in another (younger) age interval in another (earlier) period. Without loss of generality, let us for the sake of simplicity assume that time is discrete. Let at time y the vector of $\underline{x}_y$ denote the number of person-years lived prior to year y and the vector of $\underline{d}_y$ denote the increment in the number of person-years lived between time y and $y+1$. Then we will have

$$\underline{x}_y + \underline{d}_y = \underline{x}_{y+1} \tag{19}$$

If $\underline{d}_y = C \equiv const$ for all y, like in the case of constantly increasing period life expectancy at birth, then (19) reduces to

$$\underline{x}_y + C = \underline{x}_{y+1}$$

For a force of mortality of the form specified in Theorem 1, a 0.25-year annual change in the period life expectancy at birth, as in record life expectancy (Oeppen and Vaupel 2002), would mean a 0.25-age shift in terms of the equivalent person-years lived. For example the number of person-years lived between ages 60 and 61 in year y will be equal to the number of person-years lived between ages 80 and 81 eighty years later:

$${}_1L_{60}^{y} \quad \text{will be equal to} \quad {}_1L_{80}^{y+80}$$

Assuming somebody survives up to age x, in general, the formula that describes the class of Linear Shift Models will be

$${}_nL_{x-Cx}^{y} = {}_nL_{x}^{y+x} \tag{20}$$

Note that if period life expectancy rises constantly, then each of the intervals, in which we sum up person-years lived, gets increasingly shifted from the one corresponding to the original period measure (Fig. 2).

Note also that in each period the age-distribution shift increases by a value of C:

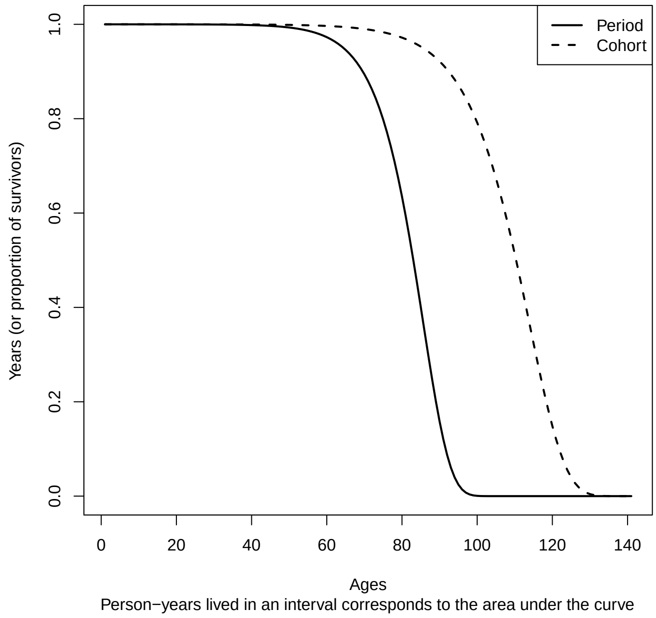

Figure 2: Person-years lived in a period t and by a cohort born in period t

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ n \end{pmatrix} + C \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 1+C \\ 2+2C \\ \vdots \\ n+nC \end{pmatrix}$$

However, a cohort experiences continuous rejuvenation due to mortality decline (see e.g., Goldstein 2008). The cohort force of mortality, μ^c, declines (shifts down) at all ages in each period.

$$\mu^c(x, y+x) = (1-r)^{y+x}\, a_0 e^{\,bx} = (1-r)^y\, a_0\, e^{(b+\ln(1-r))\,x}\,.$$

The period-by-period linearly shifting distribution of person-years lived (Eq. 20) overestimates the cohort measure. Let $\varepsilon(x, n, y)$ denote a correction term for the conversion of period based person-years lived to the person-years actually lived by a cohort (see Appendix

A). In general,

$$ {}^{c}_{n}L^{y}_{x} = \varepsilon(x,n,y)\, {}^{p}_{n}L^{y+x}_{x} = \varepsilon(x,n,y)\, {}^{p}_{n}L^{y}_{x-Cx}\,, $$

where ${}^{p}_{\cdot}L^{\cdot}_{\cdot}$ and ${}^{c}_{\cdot}L^{\cdot}_{\cdot}$ denote the period and the cohort measure, respectively.

From the definition of the cohort force of mortality, cohort life expectancy at birth e^c_0 can be computed as

$$ \begin{aligned} e^c_0(y) &= \int_0^\infty e^{-\frac{a_0}{b+\ln(1-r)}(1-r)^y\left(e^{(b+\ln(1-r))x}-1\right)}\,dx \\ &= \frac{1}{\ln(1-r)+b}\, e^{\frac{a_0(1-r)^y}{b+\ln(1-r)}} E_1\left(\frac{a_0}{b+\ln(1-r)}\,(1-r)^y\right) \end{aligned} $$

A comparison of cohort life expectancy at birth for cohorts born at time y with the period life expectancy at time y shows that

$$ \lim_{a(y)\to 0^+} \frac{e^c_0(y)}{e_0(y)} = \frac{1}{1-\left(-\frac{\ln(1-r)}{b}\right)} \tag{21} $$

$$ \tag{22} $$

Therefore,

$$ \lim_{a(y)\to 0^+} e^c_0(y) = \frac{e_0(y)}{1-\dot{e}_0(y)} \tag{23} $$

The exact relationship between period and cohort life expectancy for all y could be described in the following way:

$$e_0^c(y) \;=\; \varepsilon(0,\omega,y)\frac{e_0(y)}{1-\dot{e}_0(y)}\,,$$

Although we derived (21) by using the specific functional form of life expectancy in the Gompertz proportional hazards settings, relationship (23) holds for any Linear Shift Model. It postulates that life-expectancy of the cohort born at time y can be expressed as the period life expectancy at y, adjusted by a term that captures the shift of the mode of the mortality distribution.

Cohort Life Expectancy's Rate of Change

If period life expectancy at birth increases constantly due to a geometric decrease in the Gompertz force of mortality, the distribution of years gained by ages (i.e. change in person-years lived in an interval) shifts also by the same constant. In case of cohorts, all of the age groups experience a shift by a constant C value multiplied by the length of the respective period. As all 'period' ages get increasingly shifted in the corresponding 'cohort' mortality profile, the 'period' curve gets stretched by Cx (see Fig. 3).

The difference $\dot{e}_0^c(y)$ between two cohorts in the number of years of life expectancy gained is:

$$\dot{e}_0^c(y) \;\approx\; -\frac{a_0\ln(1-r)}{b+\ln(1-r)}(1-r)^y\int\limits_0^\infty\left(e^{(b+\ln(1-r))x}-1\right)e^{-\frac{a_0(1-r)^y}{b+\ln(1-r)}\left(e^{(b+\ln(1-r))x}-1\right)}\,dx$$

Making appropriate transformations in the integral, we get

$$\dot{e}_0^c(y) \;=\; -\frac{\ln(1-r)}{b+\ln(1-r)}\left(e^{-\frac{2a_0(1-r)^y}{b+\ln(1-r)}}+a_0(1-r)^y e_0^c(y)\right)\,.$$

In the case of changes in period life expectancy we already observed that as $\lim\limits_{y\to\infty} a_0(1-$

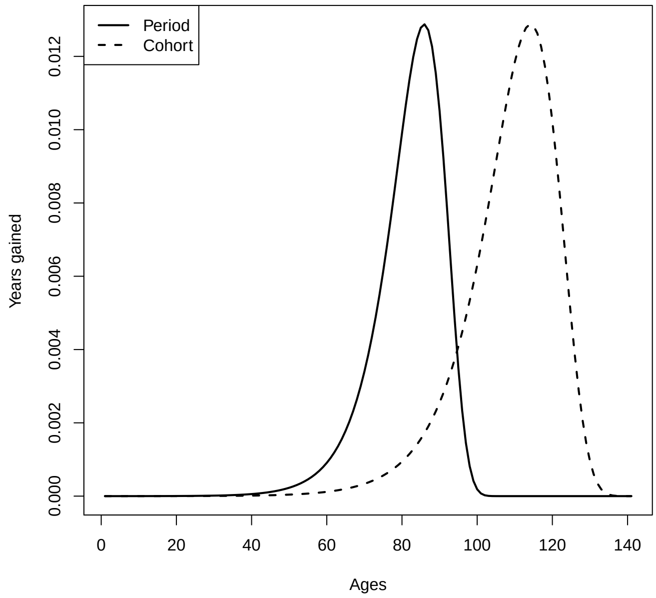

Figure 3: Distribution by ages of years gained by constant period life expectancy at birth increase at period t and cohort born in period t

$r)^y = 0$, the second additive term in the right-hand side goes to zero, the first one goes to $-\ln(1-r)/(b+\ln(1-r))$. As a result,

$$\dot{e}_0^c(y) \approx -\frac{\ln(1-r)}{b+\ln(1-r)}$$

Rearranging this to a more easily recognizable form, we get

$$\dot{e}_0^c(y) \approx \frac{-\frac{\ln(1-r)}{b}}{1-\left(-\frac{\ln(1-r)}{b}\right)}$$

As a result, when cohort life expectancy at birth e_0^c changes because of a constant rise in period life expectancy, we have

$$\dot{e}_0^c = \frac{\dot{e}_0(y)}{1 - \dot{e}_0(y)} \tag{24}$$

Thus we proved the following

Theorem 3. *Denote by* $e_0^p(y)$ *and* $e_0^c(y)$ *the period and cohort life expectancy at birth at time* y*, respectively. Suppose*

$$\dot{e}_0^p(y) = \frac{de_0^p(y)}{dy} = C \equiv const \qquad \forall t$$

Then for $a(y) \to 0+$

$$e_0^c(y) = \frac{e_0^p(y)}{1 - \dot{e}_0^p(y)} = \frac{e_0^p(y)}{1 - C} \qquad \textit{and} \qquad \dot{e}_0^c(y) = \frac{\dot{e}_0^p(y)}{1 - \dot{e}_0^p(y)} = \frac{\dot{e}_0^p(y)}{1 - C}$$

Discussion

Needless to say, the notion that there is no upper limit of life expectancy does not suggest that humans will ever be immortal. It would happen only if there were a constant proportional decrease in mortality for an infinite amount of time. Unlimited life expectancy only implies the plasticity of longevity. Humans might or might not achieve living up to currently deemed as extraordinarily high ages but there is no demographic (e.g. Wilmoth 2000; Oeppen and Vaupel 2002) or mathematical evidence that this is impossible[3].

Consider individuals that were born in 2010 in a country with a period life expectancy at birth of 80 years. In this country, assume that (i) there is a constant 2.5-year increase in life expectancy per decade and (ii) mortality improvements follow a Gompertz proportional hazards model. In a period perspective, this would mean that during their childhood and

[3]However, biodemographers would strongly disagree with it (e.g. Carnes et al 2003), claiming that "while bodies are not designed to fail, neither are they designed for extended operation" (Carnes and Olshansky 2007: 374)

young adulthood these individuals gain additional six hours of life every day (Vaupel 2009: 352). As life advances, this gain shrinks since there are less remaining years, in which mortality can improve. In order to claim that individuals gain six hours a day in their infancy, one has to assume that they will experience the same probability of death, when they turn ten in 2020, as the one pertaining to a 10-year-old child in 2010. Similarly, when they turn 80 in 2090, they would have the same probability to die as a 80-year-old person in 2010. This assumption should hold on every day or birthday of their life. However, within our assumptions, mortality improves in such a way that at that individuals aged 80 in 2090 would have the same probability to die as a 60-year-old person in 2010.

In a cohort view, when we account for the ever changing mortality regime, we can see that these life expectancy gains are not negligible. Individuals in this country will have a (cohort) life expectancy of 103.92 years. If we do not correct for overestimation by taking into account the ε term defined in Appendix A, our result will be $\frac{80}{1-0.25} = 106.67$ years. Individuals born ten years later will have a life expectancy, which will be higher by $\frac{2.5}{1-0.25} = 3.33$ years.

Life expectancy gains result from the continuous linear shift to older ages in the distribution of person-years gained. Gompertz proportional hazards models stipulate reduction in the late middle-age and old-age mortality. Mortality of oldest-old, though, stays unmodified. However, as time passes, increasingly older ages benefit from mortality reduction. Therefore, there is always space for improvement in the survival function, and life expectancy can reach infinity.

Conclusion

Gompertz proportional hazard models provide scenarios for mortality reduction in accordance with the proportionality assumption of Bongaarts and Feeney (2003). A geometrically decreasing force of mortality leads to a linearly increasing period life expectancy at birth. The linearly increasing life expectancy does not level off at any hypothetical number, its limit is infinity. This result corroborates the notion of the plasticity of longevity (e.g. Vaupel

1998). As long as mortality rates can be reduced at old and increasingly older ages, there is no upper limit of life expectancy.

A proportionally changing force of mortality leads to linearly shifting distribution of person-years gained by periods. In case of cohorts, this linear shift expands over time. Based on this relationship, cohort life expectancy at birth can be represented in terms of period life expectancy at birth and its increment. A similar representation applies to the rate of change in cohort life expectancy at birth.

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Appendix A

Linearly shifting distribution of person-years lived, $L^s(x,n,y)$, for survivors to age x can be computed as

$$
\begin{aligned}
L^s(x,n,y) &= \int\limits_x^{x+n} e^{-\frac{a_0}{b}(1-r)^y\left(e^{(b+\ln(1-r))x}-1\right)}\,dx \\
&= \frac{1}{\ln(1-r)+b}\, e^{\frac{a_0(1-r)^y}{b}}\, E\Big[^{\frac{a_0}{b}(1-r)^y\,\exp((b+\ln(1-r))(x+n))}_{\frac{a_0}{b}(1-r)^y\,\exp((b+\ln(1-r))x)}\,.
\end{aligned}
$$

This relationship shows that an artificial cohort born in year y would have lived $L^s(x,n,y)$ person-years in the interval from age x to $x+n$ if the survival rates shifted linearly. $E[^{:}_{:}$ denotes the exponential integral from the lower bound to the upper bound.

Person-years lived in the $(x, x+n]$ interval for a cohort born in year y is

$$
L^c(x,n,y) = \frac{1}{\ln(1-r)+b}\, e^{\frac{a_0(1-r)^y}{b+\ln(1-r)}}\, E\Big[^{\frac{a_0}{b+\ln(1-r)}(1-r)^y\,\exp((b+\ln(1-r))(x+n))}_{\frac{a_0}{b+\ln(1-r)}(1-r)^y\,\exp((b+\ln(1-r))x)}\,.
$$

As

$$
e^{\frac{a_0(1-r)^y}{b}} \approx e^{\frac{a_0(1-r)^y}{b+\ln(1-r)}} \approx 1\,,
$$

the difference between $L^s(x,n,y)$ and $L^c(x,n,y)$ solely lies in the bounds of the exponential integral. Let $\varepsilon(x,n,y)$ denote a correction term that transforms the linearly shifting person-years lived to the person-years lived by a cohort.

$$
\varepsilon(x,n,y) = \frac{E\Big[^{\frac{a_0}{b+\ln(1-r)}(1-r)^y\,\exp((b+\ln(1-r))(x+n))}_{\frac{a_0}{b+\ln(1-r)}(1-r)^y\,\exp((b+\ln(1-r))x)}}{E\Big[^{\frac{a_0}{b}(1-r)^y\,\exp((b+\ln(1-r))(x+n))}_{\frac{a_0}{b}(1-r)^y\,\exp((b+\ln(1-r))x)}}
$$

Correcting $L^s(x,n,y)$ with ε connects the shifted period to the cohort values:

$$
\varepsilon(x,n,y)\,L^s(x,n,y) = L^c(x,n,y)\,.
$$

This correction term is negligible in young ages. For example, if $a = 10^{-6}$, $b = 0.14$, $r = 0.034$ at

$y = 0$, in the $(10, 15]$ age interval we have

$$\varepsilon(10, 5, 0) = 0.9999 \,.$$

In the middle-age range its value is still very close to 1:

$$\varepsilon(60, 5, 0) = 0.998 \,.$$

$\varepsilon(x, n, y)$ starts to be meaningful in old ages as

$$\varepsilon(90, 5, 0) = 0.961 \,.$$

In case of (cohort) life expectancy at birth,

$$\varepsilon(0, \omega, 0) = 0.974 \,,$$

which amounts to an overestimation of 2.74 years of cohort life expectancy at birth.

Appendix B

Fig. 4 shows an example of changes in person-years lived in an age interval between two consecutive periods. We chose Japan females for demonstration. As life expectancy at birth augmentation over time is not constant even in Japan, we drew person-years lived changes from one period to another to the same scale and removed six years when life expectancy at birth was lower than in the previous year. The resulting age distribution of person-years gained and the movement of the curve resembles to the one predicted by the model (see Fig. 1).

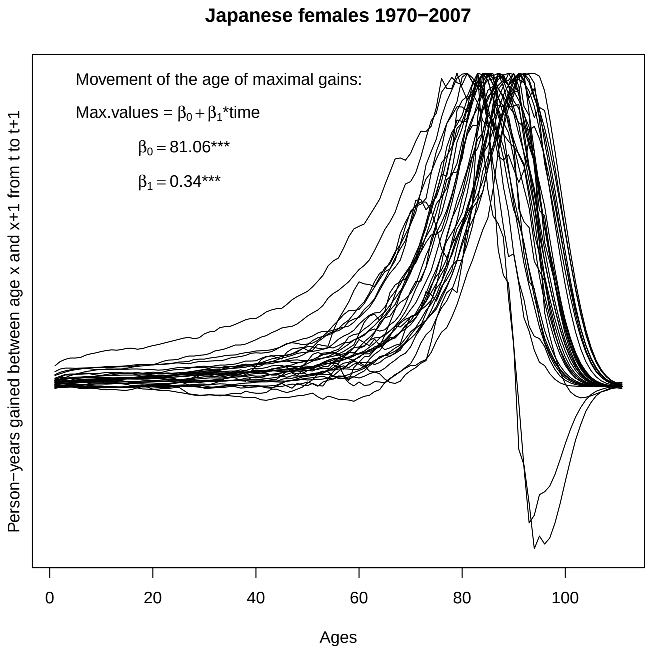

Figure 4: Distribution by ages of person-years gained